On the Discrepancy of 3 Permutations

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**Definition.** Let $X$ be a finite set, $|X| = n$. Let $A \subseteq 2^X$ a set family of subsets of $X$ and $\chi : X \rightarrow \{+1, -1\}$. Then we define:

- $\chi(S) := \sum_{a \in S} \chi(a)$
- $\text{disc}(\chi, A) := \max_{S \in A} |\chi(S)|$
- $\text{disc}(A) := \min_{\chi} \text{disc}(\chi, A) = \min_{\chi} \max_{S \in A} |\sum_{a \in S} \chi(a)|$

We transfer this concept from set families to permutations by building a hypergraph, whose edge set is a set family over the vertex set $X$:

**Definition.** Let $\pi_1, \pi_2, \ldots, \pi_m$ be permutations of the set $X$. Now we construct the edge set of a hypergraph with vertex set $X$ and the edge set:

$\Pi(\pi_1, \ldots, \pi_m) := \{\pi_i(\{p + 1, p + 2, \ldots, q\}) : 1 \leq i \leq m, 0 \leq p < q \leq n\}$.

In other words, the hypergraph has any interval of any of the permutations $\pi_i$ as a hyperedge. We will refer to the discrepancy of this hypergraph induced by the permutations as the discrepancy of the permutations $\pi_1, \ldots, \pi_m$.

### 1 Upper Bound

**Beck's Conjecture** (1987). For any set $X$ and any three permutations $\pi_1, \pi_2, \pi_3$, the discrepancy of the induced hypergraph is bounded by some constant $K$, independent of $n$.

**Proposition** (Upper Bound). (1990)

$\Pi(\pi_1, \pi_2, \ldots, \pi_m)$ has discrepancy at most $10m \cdot \log(n)$.

**Proof.** Some key steps to keep track of the proof:

1. Consider initial intervals only and double the discrepancy at the end

2. Generalize colorings to mappings $\chi : X \rightarrow [-1, +1]$, imagine the points of color $-1$ to be black, those of color $+1$ to be white and the rest to be grey

3. First, give all points color 0, then change the colors step by step

4. In each step, recolor some grey points black or white, change the shade of grey for some and leave all those already black or white in peace

5. Estimate the discrepancy after a fixed number of steps, estimate the number of steps

6. Combining these estimates gives the upper bound for the discrepancy.
2 Lower Bound

We are going to consider only permutations of length \( n = 3^k \) for \( k \in \mathbb{N} \).

Notation.

i) Denote the permutations by \( \pi_1^k, \pi_2^k \) and \( \pi_3^k \).

ii) \( \alpha^k_i(x) \) is the prefix of permutation \( \pi^k_i \), i.e. the elements 1 through \( x \) of \( \pi^k_i \), \( \alpha^k_i(0) := \emptyset \).

iii) \( \omega^k(x) \) is the suffix of permutation \( \pi^k_i \), i.e. the elements \( x \) through \( n \) of \( \pi^k_i \), \( \omega^k_i(3^k+1) := \emptyset \).

iv) \( S_k := \{ \alpha^k_i(x) \mid i = 1, 2, 3, x \in [0, 3^k] \} \).

We give a counterexample of permutations following the scheme:

\[
\begin{array}{ccc}
A & B & C \\
C & A & B \\
B & C & A
\end{array}
\]

For example, for \( k = 3 \) we get

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

Lemma 1. Given permutations \( \{ \pi^k_i \} \), we have an isomorphic relation between the three permutations induced on \( [1, 3^{k-1}], [3^{k-1} + 1, 2 \cdot 3^{k-1}] \) and \( [2 \cdot 3^{k-1} + 1, 3^k] \) and the permutations \( \{ \pi^k_{3-i} \} \).

Proposition (Lower Bound). (2011)

\[ \text{disc}(S_k) \geq \left\lceil \frac{k}{3} + 1 \right\rceil = \left\lceil \frac{\log_3 n}{3} + 1 \right\rceil \]

Notation. For a fixed coloring \( \chi \) we write

\[
\begin{align*}
\text{disc}^k_{L+}(\chi) &= \max_{x,y,z \in [0,3^k]} (\chi(\alpha^k_1(x)) + \chi(\alpha^k_2(y)) + \chi(\alpha^k_3(z))) \\
\text{disc}^k_{L}(\chi) &= \min_{x,y,z \in [0,3^k]} (\chi(\alpha^k_1(x)) + \chi(\alpha^k_2(y)) + \chi(\alpha^k_3(z))) \\
\text{disc}^k_{R+}(\chi) &= \max_{x,y,z \in [1,3^k+1]} (\chi(\omega^k_1(x)) + \chi(\omega^k_2(y)) + \chi(\omega^k_3(z))) \\
\text{disc}^k_{R}(\chi) &= \min_{x,y,z \in [1,3^k+1]} (\chi(\omega^k_1(x)) + \chi(\omega^k_2(y)) + \chi(\omega^k_3(z)))
\end{align*}
\]

Lemma 2. Let \( \Delta := \left| \chi([3^k]) \right| \). Then

\[ \begin{align*}
1 & \quad \chi([3^k]) \geq 1 \Rightarrow \text{disc}^k_{L+}(\chi), \text{disc}^k_{R+}(\chi) \geq k + \Delta + 2 \\
2 & \quad \chi([3^k]) \leq -1 \Rightarrow \text{disc}^k_{L}(\chi), \text{disc}^k_{R}(\chi) \leq -k - \Delta - 2
\end{align*} \]

Observe that Lemma 2 gives a better bound than the proposition. Moreover, Lemma 2 implies the following corollary:

Corollary. Let \( \Delta := \left| \chi([3^k]) \right| \). Then

\[ \begin{align*}
1 & \quad \chi([3^k]) \geq 1 \Rightarrow \text{disc}^k_{L+}(\chi), \text{disc}^k_{R}(\chi) \leq -k + 2\Delta - 2 \\
2 & \quad \chi([3^k]) \leq -1 \Rightarrow \text{disc}^k_{L+}(\chi), \text{disc}^k_{R+}(\chi) \geq k - 2\Delta + 2
\end{align*} \]

With the help of the corollary we can prove Lemma 2 inductively and therefore the proposition.