Masterarbeit

The Polytopes of Cardinality Homogeneous Set Systems

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0 Introduction

0.1 Overview

A Cardinality Homogeneous Set System \( C(n; a) \) is a subset of the power set \( P([n]) \) with the property that \( C(n; a) \) contains exactly those sets \( S \subseteq [n] \) with cardinality \(|S| = a_i\) for some \( i \in [m] \), where \( a = (a_1, \ldots, a_m) \) is a sequence of cardinalities. Each such set system has an associated polytope \( P(n; a) \), which is the convex hull of the set of incidence vectors of \( C(n; a) \). This work will give an analysis of the faces of the polytope based on the description of \( P(n; a) \) as an \( \mathcal{H} \)-polytope worked out by Grötschel (2004). A classification of all its faces regarding dimension and combinatorial type will be provided and the f-vector of \( P(n; a) \) will be deduced.

In the first section, »Set Systems and Polytopes«, we introduce the objects to be studied and some basic polytope theory related to them. The second section, »A Halfspace Description«, provides some results of Grötschel (2004) concerning a complete description of the polytope in terms of hyperplanes bounding it. Based on these results, we will establish our course of action for the subsequent sections. The third section, »Vertices on Hyperplanes«, describes the sets of vertices of the polytope on each of the hyperplanes and some observations about the vertices contained in the intersection of certain classes of hyperplanes. In the fourth section, »A Classification of the Faces«, we will consider any possible intersection of hyperplanes, determine the associated vertex set and some more properties. For this purpose, we will establish a detailed classification of the faces. The fifth section, »Counting the Faces«, uses our previous findings to deduce the f-vector of the polytope.

0.2 Motivation

Some combinatorial problems can be easily expressed in terms of polytopes. At times, these translations facilitate more efficient solving algorithms when operating on the associated geometrical objects. A better understanding of them could therefore allow for more efficient solutions to a combinatorial problem associated with a cardinality homogeneous set system. Grötschel (2004) showed that the cycle and circuit polytopes associated with uniform matroids, which are of interest for the solution of combinatorial optimisation problems, are in fact polytopes of cardinality homogeneous set systems.

0.3 Notation

Let us start with some remarks on the notation used below. We will make ample use of binomial coefficients, where \( \binom{n}{k} \) is equal to the coefficient of \( X^k \) in \((1 + X)^n\), a definition allowing us to use binomial coefficients with negative \( k \), yielding \( \binom{n}{-k} = 0 \) as there is no term of the form \( X^k \) with negative \( k \) in \((1 + X)^n\). We will also use the short binomial notation for the set of subsets of a certain cardinality:

\[
\binom{A}{k} = \{ B \subseteq A : |B| = k \}
\]

For any set \( A \), its power set, i.e. the set containing all subsets of \( A \), shall be denoted by \( \mathcal{P}(A) \). We will work a lot on both sets \( S \subseteq [n] \) and points \( x \in \mathbb{R}^n \). We try to avoid any confusion by using upper case letters for sets and lower case letters for points. The lower case letters \( i, j, k, \ell \) are reserved for use as indices.

We will use 0 and 1 to indicate the all-zeros and the all-ones-vector, respectively. \( e_i \) shall indicate the unit vector in direction \( i \). If not indicated otherwise, any vector shall be a column
vector, and we will use $v^\top$ to indicate the transposed version, i.e. the corresponding row vector, whenever necessary. $I_n$ shall indicate the $n$-dimensional identity matrix.

Established definitions will be written in the clinical form »X is Y«. The definitions we introduce ourselves will be recognisable by formulations such as »We shall call« or »We define«.

1 Set Systems and Polytopes

In this section, we will introduce the objects we will be working on henceforth: Cardinality homogeneous set systems and their polytopes, as introduced in (Grötschel, 2004). In addition to introducing the concept of cardinality homogeneity in terms of sets, we will introduce some basic polytope theory. For this, we both follow the definitions and use the propositions given in (Ziegler, 1995).

1.1 Cardinality Homogeneous Set Systems

Let us take a look at what we are dealing with.

Definition 1 (Cardinality Homogeneous Set System). Let $n \in \mathbb{N}$ be a positive integer and $[n]$ be the set of all positive integers from 1 to $n$. A set system $C \subseteq \mathcal{P}([n])$ is cardinality homogeneous if, whenever $C$ contains some subset of cardinality $k$ with $0 \leq k \leq n$, then $C$ contains all subsets of $[n]$ of cardinality $k$.

To get a grip of this notion, let us have a look at some examples as listed by Grötschel (2004):

Example 2. The following set systems are cardinality homogeneous.

1. $C = \mathcal{P}([n])$, the set of all subsets of $[n]$;
2. $C = \{S \subseteq [n] : |S| \text{ is even} \}$;
3. $C = \{S \subseteq [n] : |S| \text{ is odd} \}$.

Grötschel (2004) introduced the cardinality sequence and the set $C(n; a)$ in order to be able to describe cardinality homogeneous set systems concisely. We add the concept of gaps and layers to facilitate speaking about sets $S \subseteq [n]$ of certain cardinalities.

Definition 3 (Cardinality Sequence). Let $n, m \in \mathbb{N}$ with $1 \leq m \leq n$. A cardinality sequence for $n$ is a sequence $a = (a_1, \ldots, a_m)^\top$ of strictly increasing integers $0 \leq a_1 < a_2 < \cdots < a_m \leq n$.

Definition 4 ($C(n; a)$). Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. The cardinality homogeneous set system $C(n; a)$ is defined as follows:

$$C(n; a) := \bigcup_{k=1}^m \binom{[n]}{a_k}.$$ 

Definition 5 (Gap and layer). Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. We say that $S \subseteq [n]$ lies in gap $g \in [m-1]$, if $a_g < |S| < a_{g+1}$. We extend this definition to say that $S$ lies in gap $0$ if $|S| < a_1$ or $S$ lies in gap $m$ if $a_m < |S|$. If $|S| = a_\ell$, we say that $S$ lies on layer $\ell$. 

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Figure 1: A Cardinality Homogeneous Set System with base set [5]

Clearly, any cardinality homogeneous set system is of the form $C(n; a)$ for some $n$ and $a$. We will continue working with this canonical form. But first, let us use an example to display the concept introduced so far. We choose $n = 5$, $a = (1, 4)$ and take a look at the associated cardinality homogeneous set system in Figure 1.

Figure 1a) illustrates a cardinality homogeneous set system on the Boolean lattice of $[5]$: Sets contained in $C(5; (1, 4))$, i.e. subsets of $[5]$ with cardinality 1 or 4, are drawn in grey. Figure 1b) illustrates the definition of gap and layer: The sets in gap 0 are coloured in orange, the sets on layer 1 are blue, those in gap 1 are yellow, the sets on layer 2 are green and those in gap 2 are coloured in red.

### 1.2 The Associated Polytopes

Although we now know the class of cardinality homogeneous set systems, it is not at all clear yet what their polytopes are. In the previous section, we only talked about sets and their cardinalities, whereas polytopes are geometrical objects in space. So, first and foremost, we need some sort of instructions as to how one can turn a set system into a polytope.

A well-known concept to translate things defined in terms of sets into geometrical objects is that of incidence vectors:

**Definition 6 (Incidence vector).** Given a base set $[n]$, the incidence vector of a subset $S \subseteq [n]$ is the $n$-dimensional vector $\chi_S \in \{0, 1\}^n$ with $\chi_S = (\chi_S(1), \chi_S(2), \ldots, \chi_S(n))$ and entries defined in the following way:

$$\chi_S(i) := \begin{cases} 1, & \text{if } i \in S \\ 0, & \text{if } i \notin S. \end{cases}$$

Incidence vectors thus give us a way to regard a single subset of $[n]$ as a single point in $\mathbb{R}^n$. To regard an entire cardinality homogeneous set system as a geometric object in $\mathbb{R}^n$, we need some additional notions:

**Definition 7 (Affine and convex hull).** Let $X$ be a finite set of points in $\mathbb{R}^n$. The affine hull $\text{aff}(X)$ is the intersection of all affine subspaces of $\mathbb{R}^n$ that contain $X$. The convex hull $\text{conv}(X)$ is the intersection of all convex subsets of $\mathbb{R}^n$ that contain $X$. 


As any affine subspace is convex, so are their intersections, therefore the convex hull of a finite set of points is a subset of its affine hull.

We may now assemble the points resulting from the translation achieved by incidence vectors into a single object by constructing their convex hull, yielding a $V$-polytope:

**Definition 8 ($V$-polytope).** A $V$-polytope is the convex hull of a finite set of points in some $\mathbb{R}^d$.

As a cardinality homogeneous set system $C(n; a)$ contains finitely many elements and each of the contained sets has a uniquely defined incidence vector, the set of their incidence vectors is a finite point set in $\mathbb{R}^n$, so their convex hull is a $V$-polytope, therefore the following is well-defined:

**Definition 9 (Polytope of a Cardinality Homogeneous Set System).** Let $n \in \mathbb{N}$ with a cardinality sequence $a$. Let $C(n; a)$ be the associated cardinality homogeneous set system. Then

$$P(n; a) := \text{conv}\{\chi_S : S \in C(n; a)\}$$

is called a **polytope of a cardinality homogeneous set system**.

We first note that we may describe the polytopes directly from $n$ and $a$ by observing that

$$P(n; a) = \text{conv}\{\chi_S : S \in C(n; a)\} = \text{conv}\{\chi_S : S \in \bigcup_{k=1}^{m} \binom{[n]}{a_k}\} = \text{conv}\{x \in \{0, 1\}^n : 1^\top x \in \{a_1, \ldots, a_m\}\}.$$

We already used something very similar to this scalar product with the all-ones-vector in terms of subsets of $[n]$ when we defined gaps and layers. We observe that a subset $S$ of $[n]$ lies in gap $i \in [m-1]$ if and only if $\chi_S$ lies in the intersection of the open halfspaces $\{x \in \mathbb{R}^n : 1^\top x > a_i\} \cap \{x \in \mathbb{R}^n : 1^\top x < a_{i+1}\}$. Furthermore, we observe that $P(n; a)$ is a subset of the $n$-dimensional cube $[0, 1]^n$.

### 1.3 The Vertices of $P(n; a)$

So far, we have only defined $P(n; a)$ to be the convex hull of the incidence vectors of the sets in $C(n; a)$ – we have not yet stated anything about the meaning these sets bear in the resulting polytope. In order to do so, we need to introduce some of the most basic notions in polytope theory.

**Definition 10 (Face of a polytope).** Let $P \subseteq \mathbb{R}^d$ be a convex polytope. A linear inequality $c^\top x \leq c_0$ is **valid** for $P$ if it is satisfied for all points $x \in P$. A **face** of $P$ is any set of the form

$$F = P \cap \{x \in \mathbb{R}^d : c^\top x = c_0\}$$

where $c^\top x \leq c_0$ is a valid inequality for $P$. A face $F$ which is not equal to $P$ is called a **proper** face. The affine hull of a face, $\text{aff}(F)$, is the intersection of all affine subspaces of $\mathbb{R}^d$ that contain $F$. The **dimension** of a face is the dimension of its affine hull:

$$\dim(F) := \dim(\text{aff}(F)).$$

The faces of dimension 0 are called **vertices**. The set of all vertices of $P$ is called the **vertex set** of $P$, denoted by $\text{vert}(P)$. 
Let us now walk our way through these definitions with a little example on our convex polytope \( P(n; a) \subseteq \mathbb{R}^n \). We fix an arbitrary set \( S \in C(n; a) \) and define \( c(S) := 2\chi_S - 1 \). Now \( c(S) \) is an \( n \)-dimensional vector with entries \( c(S)_i = 1 \) where \( i \in S \) and \( c(S)_i = -1 \) where \( i \notin S \). If we now define \( c(S)_0 := |S| \), we see that \( c(S)^\top x \leq c(S)_0 \) is equivalent to \( \sum_{i \in S} x_i - \sum_{i \notin S} x_i \leq |S| \). This holds for any \( x \in [0, 1]^n \) and consequently for \( P(n; a) \), which is a subset thereof. Moreover, it is clear that \( P(n; a) \cap \{ x \in \mathbb{R}^n : c(S)^\top x = c(S)_0 \} = \{ \chi_S \} \), therefore \( \{ \chi_S \} \) is a face of \( P(n; a) \).

If \( S \) is the only element of \( C(n; a) \), then \( \{ \chi_S \} \) is equal to \( P(n; a) \), otherwise, it is a proper face of it. It is an affine subspace of \( \mathbb{R}^n \), so it is equal to its affine hull and \( \dim(\{ \chi_S \}) = 0 \). This implies that it is a vertex of \( P(n; a) \). As we chose \( S \) arbitrarily from \( C(n; a) \), we may conclude that \( \{ \chi_S : S \in C(n; a) \} \subseteq \text{vert}(P(n; a)) \).

The following statement is a concatenation of Propositions 2.2 and 2.3 in \( \textit{Ziegler, 1995} \).

**Proposition 11.** Let \( P \subseteq \mathbb{R}^d \) be a polytope, \( V := \text{vert}(P) \) and \( F \) be a face of \( P \). Then:

1. Every polytope is the convex hull of its vertices: \( P = \text{conv}(V) \).
2. If a polytope can be written as the convex hull of a finite point set, then the set contains all the vertices of the polytope: \( P = \text{conv}(V) \) implies that \( \text{vert}(P) \subseteq V \).
3. The face \( F \) is a polytope, with \( \text{vert}(F) = F \cap V \).
4. Every intersection of faces of \( P \) is a face of \( P \).
5. The faces of \( F \) are exactly the faces of \( P \) that are contained in \( F \).

First of all, we may apply Proposition 112 to the polytopes \( P(n; a) \) defined to be the convex hull of \( \{ \chi_S : S \in C(n; a) \} \) to find out that \( \text{vert}(P(n; a)) \subseteq \{ \chi_S : S \in C(n; a) \} \). We may combine this with the complementary inclusion we argued above, and conclude the following:

**Observation 12.** Let \( n \in \mathbb{N} \) and \( a \) be a cardinality sequence for \( n \). Then:

\[
\text{vert}(P(n; a)) = \{ \chi_S : S \in C(n; a) \}.
\]

The other statements contained in Proposition 11 will be used later. For now, we turn away from vertices and convex hulls to see the polytopes of cardinality homogeneous set systems from another perspective. There is another canonical way to define a polytope:

**Definition 13** (\( \mathcal{H} \)-polytope). An \( \mathcal{H} \)-polytope is a bounded intersection of finitely many closed halfspaces in some \( \mathbb{R}^d \).

This leads us to cite the following main theorem for polytopes, claiming that \( \mathcal{H} \)- and \( \mathcal{V} \)-polytopes are in fact the same objects, described in different ways:

**Theorem 14** (Main Theorem for Polytopes). A subset \( P \subseteq \mathbb{R}^d \) is the convex hull of a finite point set (a \( \mathcal{V} \)-polytope) if and only if it is a bounded intersection of finitely many halfspaces (an \( \mathcal{H} \)-polytope).

Thanks to this statement, it is clear that every \( \mathcal{V} \)-polytope, especially the polytope of a cardinality homogeneous set system as defined above, has a representation in terms of intersected halfspaces. In the process of examining \( P(n; a) \), it is now a very natural question to ask which halfspaces one needs to intersect in order to create \( P(n; a) \). This question was answered thoroughly in \( \textit{Grötschel, 2004} \). In the next section, we will review these findings before using them to further analyse the polytopes of cardinality homogeneous set systems.
2 A Halfspace Description

In this section, we will try to find a representation of $P(n; \alpha)$ in terms of an intersection of finitely many halfspaces. We firstly observe that any halfspace containing $P(n; \alpha)$ is defined by an inequality which is necessarily valid for $P(n; \alpha)$ and therefore valid for $\text{vert}(P(n; \alpha))$. Let us start by listing some valid inequalities.

2.1 Some Inequalities for $P(n; \alpha)$

Let us recall that all the vertices of $P(n; \alpha)$ are points in $\{0, 1\}^n$. Thus, the inequalities $0 \leq x_i \leq 1$ both hold.

**Definition 15** (0/1-Inequality). Let $n \in \mathbb{N}$. For all $i \in [n]$, the inequality $x_i \geq 0$ shall be called the **0-inequality** associated with $i$, while the inequality $x_i \leq 1$ shall be called the **1-inequality** associated with $i$.

Furthermore, we recall that any set $S$ in $C(n; \alpha)$ has a cardinality listed in the cardinality sequence $\alpha$, which implies that $a_1 \leq |S| \leq a_m$. In combination with $|S| = \sum_{i=1}^n x_S$, this yields $a_1 \leq \sum_{i=1}^n x_i \leq a_m$ for all vertices of $P(n; \alpha)$.

**Definition 16** (min/max-Inequality). Let $n \in \mathbb{N}$ and $\alpha$ be a cardinality sequence for $n$. The inequality $\sum_{i=1}^n x_i \leq a_m$ shall be called the **min-inequality**, while the inequality $\sum_{i=1}^n x_i \leq a_m$ shall be called the **max-inequality**.

Grötschel (2004) classified another type of inequalities associated with $P(n; \alpha)$. He named them cardinality forcing-inequalities (or CF-inequality for short), as each of them separates the incidence vector of a set of unwanted cardinality from the incidence vectors of the sets in $C(n; \alpha)$. One such inequality is introduced for each set of a cardinality $k$ with $a_1 < k < a_m$ not listed in the cardinality sequence. Let us first formally introduce the collection of unwanted sets, before expressing the CF-inequality as introduced by Grötschel (2004).

**Definition 17** (CF-Inequality). Let $n \in \mathbb{N}$ and $\alpha$ be a cardinality sequence for $n$. We define

$$S(n; \alpha) := \{ S \subset [n] : a_1 < |S| < a_m, |S| \notin \{a_1, \ldots, a_m\} \}.$$  

For $S \in S(n; \alpha)$, $S$ in gap $g$, the associated **CF-inequality** is

$$\sum_{i \in S} (a_{g+1} - |S|) x_i - \sum_{i \in [n] \setminus S} (|S| - a_g) x_i \leq (a_{g+1} - |S|) a_g.$$  

We note that all the inequalities defined above may be expressed in the canonical form $c^T x \leq c_0$:

- The 0-inequality $x_i \geq 0$ is equivalent to $-e_i^T x \leq 0$
- The 1-inequality $x_i \leq 1$ is equivalent to $e_i^T x \leq 1$
- The min-inequality $\sum_{i=1}^n x_i \leq a_m$ is equivalent to $-1^T x \leq -a_1$
- The max-inequality $\sum_{i=1}^n x_i \leq a_m$ is equivalent to $1^T x \leq a_m$
- The CF-inequality $\sum_{i \in S} (a_{g+1} - |S|) x_i - \sum_{i \in [n] \setminus S} (|S| - a_g) x_i \leq (a_{g+1} - |S|) a_g$ is equivalent to $((a_g - |S|) 1^T + (a_{g+1} - a_g)x_S^T) x \leq (a_{g+1} - |S|) a_g$. 

2.2 The $\mathcal{H}$-Polytope $Q(n; a)$

We will now encode all these inequalities into an $\mathcal{H}$-polytope: The solution set of any linear inequality is a closed halfspace in $\mathbb{R}^n$. As we listed only finitely many of them and any set of points satisfying both $x_i \geq 0$ and $x_i \leq 1$ for every direction $i$ is evidently bounded, the intersection of all the halfspaces is an $\mathcal{H}$-polytope. Additionally, the set of points that satisfy one of the inequalities with equality (i.e., the boundary of the associated halfspace) will be properly defined as well.

We can use the canonical form of the aforementioned inequalities to list them in a matrix $A$ and a vector $z$ such that a point $x \in \mathbb{R}^n$ is contained in all of the halfspaces if and only if $Ax \leq z$ holds. We will list the inequalities in the same order in which we defined them, starting with the 0/1-inequalities.

**Definition 18** ($Q(n;a)$). Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. We set $s := |S(n; a)|$ and $r := 2n + 2 + s$, then define $A(n; a) \in \mathbb{R}^{r \times n}$ and $z(n; a) \in \mathbb{R}^r$:

$$A(n; a) := \begin{pmatrix} -I_n & 0 \\ I_n & 1 \\ -1^T & -a_1 \\ 1^T & a_m \\ B \end{pmatrix} =: z(n; a),$$

where the matrix $B \in \mathbb{R}^{s \times n}$ consists of the row vectors $((a_g - |S|)1^T + (a_{g+1} - a_g)\chi_S^1)$, listed along shortlex order of all the $S \in S(n; a)$, and the vector $b \in \mathbb{R}^s$ lists the associated scalars $(a_{g+1} - |S|)a_g$. We define the $\mathcal{H}$-polytope $Q(n; a)$ associated with these inequalities:

$$Q(n; a) := \{ x \in \mathbb{R}^n : A(n; a)x \leq z(n; a) \}.$$  

In addition to this result of inequalities, for every row index $i \in [r]$ of the matrix $A(n; a)$, we define the hyperplane spanned by the associated equality and the collection of these hyperplanes:

$$\text{eq}(n; a; i) := \{ x \in \mathbb{R}^n : e_i^T A(n; a)x = e_i^T z(n; a) \};$$

$$\mathcal{H}(n; a) := \{ \text{eq}(n; a; i) : i \in [r] \}.$$  

The first half of definitions is just necessary to give a formal definition of $Q(n; a)$. We recognize that the first $2 + 2n$ rows of $A(n; a)$ and $z(n; a)$ are equal to the inequalities in canonical form. Afterwards, we used shortlex ordering (that is, an ordering of the sets primarily by ascending cardinality, then by lexicographical order) to give a well-defined form of the rest of the matrix containing the CF-inequalities.

The last part uses the structure provided by $A(n; a)$ and $z(n; a)$ to define the set of points in $\mathbb{R}^n$, where the $i$-th inequality holds with equality: $e_i^T A(n; a)$ is equal to the row vector associated with the $i$-th inequality, while $e_i^T z(n; a)$ gives the respective scalar on the right-hand side. Thus, $\text{eq}(n; a; i)$ defines the hyperplane spanned by this equality and their collection $\mathcal{H}(n; a)$ together with the ordering of inequalities will prove to be helpful for indicating which of the hyperplanes a point lies on.

Grötschel (2004) (p. 105) introduces this $\mathcal{H}$-polytope $Q(n; a)$ in a less formal way, albeit completely equivalent to our definition. He then uses the theory of linear programming and a specifically designed algorithm to prove that $P(n; a)$ is equal to the intersection of all the halfspaces associated with the above inequalities.
Proposition 19. Let \( n \in \mathbb{N} \) and \( a \) be a cardinality sequence for \( n \). Then:

\[
P(n; a) = Q(n; a).
\]

Grötschel (2004) continues by discussing the circumstances under which one of the inequalities defines a facet (that is, an inclusion-maximal proper face) of \( P(n; a) \). We are, however, aiming to find faces of all dimensions. Therefore, let us finish reviewing Grötschel’s findings now that we have obtained a description of \( P(n; a) \) as an \( \mathcal{H} \)-polytope and try to develop our own course of action to find all the faces of \( P(n; a) \).

2.3 Faces Defined by Hyperplanes

We start by stating that the intersection of the polytope with any number of hyperplanes of our three classes 0/1, min/max and CF is a face of the polytope.

Proposition 20. Let \( n \in \mathbb{N} \) and \( a \) be a cardinality sequence for \( n \). Let \( H \subseteq \mathcal{H}(n; a) \) be a set of hyperplanes. Then there is a face \( F \) of \( P(n; a) \) such that \( F \) is the intersection of \( P(n; a) \) with the hyperplanes in \( H \).

Proof. Let \( H \subseteq \mathcal{H}(n; a) \) be a set of hyperplanes. If \( H = \emptyset \), then \( F = P(n; a) \) is a face equal to the intersection of \( P(n; a) \) with no hyperplanes.

If \( H \neq \emptyset \), let us define \( F(H) := \bigcap_{h \in H}(P(n; a) \cap h) \). We set \( A := A(n; a) \) and \( z := z(n; a) \). As \( Q(n; a) \) was defined to contain all points \( x \in \mathbb{R}^n \) with \( Ax \leq z \), for all \( e_i(n; a; i) = h \in H \), the inequality \( e_i^T Ax \leq e_i^T z \) is valid for \( Q(n; a) \). By Proposition 19 it is valid for \( P(n; a) \) as well.

Therefore, for all \( e_i(n; a; i) = h \in H \), the set \( P(n; a) \cap \{x \in \mathbb{R}^n : e_i^T Ax = e_i^T z\} = P(n; a) \cap h \) is a face of \( P(n; a) \). Thus \( F(H) \) is the intersection of faces of \( P(n; a) \) and as such, by Prop. 11, a face of \( P(n; a) \).

We saw the implication of any intersection of hyperplanes being a face of our polytope. We will now examine the opposite direction: Given a face of the polytope, is it equal to some intersection of the hyperplanes? We will state and prove that this is indeed the case. The combination of the two directions paves the way for our approach to find all the faces of \( P(n; a) \) by giving us an if-and-only-if property shared by all of them.

Proposition 21. Let \( n \in \mathbb{N} \) and \( a \) be a cardinality sequence for \( n \). Let \( F \) be a face of \( P(n; a) \). Then there is a set of hyperplanes \( H \subseteq \mathcal{H}(n; a) \) such that \( F \) is equal to the intersection of \( P(n; a) \) with the hyperplanes in \( H \).

To prove this statement, we will use the Farkas lemma, a statement known in numerous equivalent formulations, which is fundamental to polytope theory, concerning the solvability of a system of inequalities. The following formulation is adequate to our problem and appears as Proposition 1.9 in Ziegler (1995), where the reader is referred to for its proof.

Proposition 22. Let \( A \in \mathbb{R}^{r \times n} \), \( z \in \mathbb{R}^r \), \( b \in \mathbb{R}^n \) and \( z_0 \in \mathbb{R} \). Then \( b^\top x \leq z_0 \) is valid for all \( x \in \mathbb{R}^n \) with \( Ax \leq z \), if and only if

1. there exists a vector \( c \geq 0 \) such that \( c^\top A = b^\top \) and \( c^\top z \leq z_0 \), or
2. there exists a vector \( c \geq 0 \) such that \( c^\top A = 0^\top \) and \( c^\top z < 0 \),

or both.
We may now define a set of variables to meet the requirements of the Farkas lemma, and then apply it to prove our proposition.

**Proof of Proposition 21.** We first observe that for \( F = P(n; a) \), we may choose \( H := \emptyset \), as \( F \) is equal to the intersection of \( P(n; a) \) and none of the hyperplanes. For \( F = \emptyset \), we may set \( H := \{ \text{eq}(n; a; 1), \text{eq}(n; a; n + 1) \} \), the intersection of the associated hyperplanes being \( \{ x \in \mathbb{R}^n : -x_1 = 0 \} \cap \{ x \in \mathbb{R}^n : x_1 = 1 \} = \emptyset \).

Let us now consider a non-empty proper face \( F \). By definition of a face, there are \( b \in \mathbb{R}^n, z_0 \in \mathbb{R} \) such that \( b^\top x \leq z_0 \) is valid for \( P(n; a) \) and \( F = P(n; a) \cap \{ x \in \mathbb{R}^n : b^\top x = z_0 \} \). We note that \( b = 0 \) would yield either \( F = P(n; a) \) (if \( z_0 \geq 0 \)) or \( F = \emptyset \) (if \( z_0 < 0 \)), so we conclude that \( b \neq 0 \).

We set \( A = A(n; a) \in \mathbb{R}^{r \times n} \) and \( z = z(n; a) \in \mathbb{R}^r \), where \( r = 2 + 2n + |S(n; a)| \). Prop. 19 tells us that \( P(n; a) = \{ x \in \mathbb{R}^n : Ax \leq z \} \). Due to our construction, \( P(n; a) \neq \emptyset \), so there is an \( x \in \mathbb{R}^n \) with \( Ax \leq z \). Therefore, there cannot be any vector \( c \geq 0 \) with \( c^\top A = 0, c^\top z < 0 \) as this would imply the contradiction \( 0 = 0x = c^\top Ax \leq c^\top z < 0 \).

With this in mind, the Farkas lemma tells us that there is a vector \( c \geq 0 \) with \( c^\top A = b^\top \) and \( c^\top z \leq z_0 \). Let us first handle the possibility that there is such a vector with \( c^\top z < z_0 \): In this case, for every point \( x \in P(n; a) \), \( Ax \leq z \) implies that \( b^\top x = c^\top Ax \leq c^\top z < z_0 \). Thus, \( F = P(n; a) \cap \{ x \in \mathbb{R}^n : b^\top x = z_0 \} = \emptyset \), in contradiction to \( F \) being a non-empty proper face.

We thereby learn that we may fix a vector \( c \geq 0 \) with \( c^\top A = b^\top \) and \( c^\top z = z_0 \). Let us note that \( c 
eq 0 \), as otherwise \( c^\top A = 0 \neq b^\top \). Let \( C := \{ i \in [r] : c_i > 0 \} \neq \emptyset \) indicate the set of positive entries of \( c \) and \( H := \{ \text{eq}(n; a; i) : i \in C \} \) the set of hyperplanes associated with the rows of \( A \) listed in \( C \). We claim that \( F \) is the intersection of \( P \) with the hyperplanes in \( H \).

To prove this claim, we pick an arbitrary \( x \in F \). It is clear by the definition of a face that \( x \in P(n; a) \). Assume for a contradiction that there is an \( i \in C \) such that \( x \notin \text{eq}(n; a; i) \). As for all \( j \in C \), the inequality \( e_i^\top A x \leq e_i^\top z \) is valid by definition of \( A \) and \( z \), we conclude that \( e_i^\top A x < e_i^\top z \) and consequently, \( b^\top x = c^\top Ax < c^\top z = z_0 \) in contradiction to \( x \in F \).

On the other hand, if \( x \) is contained in \( P(n; a) \) and in all of the hyperplanes in \( H \), then for every \( i \in C \), the inequality \( e_i^\top A x = e_i^\top z \) holds and consequently, \( b^\top x = c^\top Ax = c^\top z = z_0 \), implying that \( x \in F \) and concluding the proof of the claim.

Now that we know every face of \( P(n; a) \) to be an intersection of \( P(n; a) \) and some hyperplanes in \( H(n; a) \) (Prop. 21) and conversely, every such intersection to be a face of \( P(n; a) \) (Prop. 20), we see the opportunity to find all the faces of \( P(n; a) \) just by considering all such intersections.

We recall Proposition 11.3 telling us that for every polytope \( P \), each of its faces \( F \subseteq P \) is a polytope itself with vertex set \( \text{vert}(F) = F \cap \text{vert}(P) \). With the help of Proposition 11.1 adding the fact that \( F = \text{conv}(\text{vert}(F)) \), we conclude that \( F = \text{conv}(F \cap \text{vert}(P)) \).

In the case of an intersection of \( P(n; a) \) with all the hyperplanes in some non-empty collection \( H \subseteq H(n; a) \), we now know the associated face to be equal to \( \text{conv}(\bigcap_{h \in H} (\text{vert}(P(n; a)) \cap h)) \). This means for us that, going through all possible intersections among the sets of vertices on a hyperplane in \( H(n; a) \), the resulting vertex sets are exactly the vertex sets of the faces of \( P(n; a) \).

We have therefore found a course of action: Firstly, for each inequality of the three types defined above, we will examine which vertices of \( P(n; a) \) lie on the associated hyperplane (i.e. satisfy the inequality with equality). Secondly, we will consider the intersections of all possible combinations of these vertex sets and bring to light some of the properties the associated face (that is, the convex hull of this vertex set) has.

Will will see a formal proof, granting the accuracy of this approach, later on, when we discuss the faces. For now, we will define some handy objects we can work with to examine the vertices.
on the hyperplanes.

3 Vertices on Hyperplanes

Although the hyperplanes we consider are objects in n-dimensional space and the sets are objects defined with respect to a set \([n]\), we know that the vertices of \(P(n; a)\) correspond to the sets in \(C(n; a)\) (they are exactly their incidence vectors). For ease of notation, let us just call the sets in \(C(n; a)\) «vertices», although it is their incidence vectors that are the actual vertices.

We want to find out which sets in \(C(n; a)\) have incidence vectors lying on a particular hyperplane defined by the three classes of inequalities discussed previously and used to define the \(H\)-representation of \(P(n; a)\). In the same manner we defined the inequalities, we will now define the vertex sets containing exactly those elements of \(C(n; a)\) having incidence vectors satisfying the inequalities with equality.

**Definition 23 (Vertices on a hyperplane).** Let \(n \in \mathbb{N}\) and \(a\) be a cardinality sequence for \(n\). For every \(i \in [n]\), we define:

\[
V_0(n; a; i) := \{X \in C(n; a) : \chi_X(i) = 0\} \quad \text{and} \quad V_1(n; a; i) := \{X \in C(n; a) : \chi_X(i) = 1\}.
\]

Furthermore, we define:

\[
V_B(n; a; \min) := \{X \in C(n; a) : \sum_{i=1}^{n} \chi_X(i)\} \quad \text{and} \quad V_B(n; a; \max) := \{X \in C(n; a) : \sum_{i=1}^{n} \chi_X(i) = a_m\}.
\]

Finally, for every \(S \in S(n; a)\), depending on the gap \(g\) that \(S\) is in, we define:

\[
V_{CF}(n; a; S) := \{X \in C(n; a) : \sum_{i \in S}(a_{g+1} - |S|)\chi_X(i) - \sum_{i \in [n]\setminus S}(|S| - a_g)\chi_X(i) = (a_{g+1} - |S|)a_g\}.
\]

For each of the three classes \((0/1, \min/\max, CF)\), we will now determine which sets in \(C(n; a)\) lie on the respective hyperplane equal to the boundary of the halfspace which is the solution set of the inequality. We will proceed in the established succession \(0/1 – \min/\max – CF\).

3.1 Vertices on 0/1-Hyperplanes

**Proposition 24.** Let \(n \in \mathbb{N}\) and \(a\) be a cardinality sequence for \(n\). For every \(i \in [n]\),

\[
V_0(n; a; i) = \bigcup_{j=1}^{m} \{X \in \binom{[n]}{a_j} : i \not\in X\} \quad \text{and} \quad V_1(n; a; i) = \bigcup_{j=1}^{m} \{X \in \binom{[n]}{a_j} : i \in X\}.
\]
Proof. Let \( X \in C(n;a) \), the set of all subsets of \([n]\) of a cardinality listed in the cardinality sequence \( a \). Then \( V_0(n; a; i) = \{X \in C(n; a) : \chi_X(i) = 0\} = \{X \in C(n; a) : i \not\in X\} = \bigcup_{j=1}^m \{X \in \binom{[n]}{a_j} : i \not\in X\} \) and \( V_1(n; a; i) = \{X \in C(n; a) : \chi_X(i) = 1\} = \{X \in C(n; a) : i \in X\} = \bigcup_{j=1}^m \{X \in \binom{[n]}{a_j} : i \in X\} \).

In Figure 2 we use our examplary cardinality homogeneous set system \( C(5; (1, 4)) \) introduced in the beginning to illustrate some vertex sets on 0/1-hyperplanes. In both diagrams, we depict the vertices included in the set in black, while the other vertices of the cardinality homogeneous set system are coloured in grey and vertices of cardinalities not listed in \( a \) are white.

![Figure 2: Vertices of \( C(5; (1, 4)) \) on some 0/1-hyperplanes](image)

Let us now see what the intersection of multiple 0/1-hyperplanes looks like. For this purpose, we define sets collecting the indices of the hyperplanes of either type which we want to intersect. We shall call \( \mathbb{I} \subseteq [n] \) the in-set, collecting those indices for which we want to intersect with the 1-hyperplane, and \( \emptyset \subseteq [n] \) the out-set, collecting those indices for which we want to intersect with the 0-hyperplane.

As a little example, if we are examining some polytope \( P(n; a) \) and want to intersect it with the hyperplanes \( \{x \in \mathbb{R}^n : x_1 = 0\}, \{x \in \mathbb{R}^n : x_4 = 0\} \) and \( \{X \in \mathbb{R}^n : x_2 = 1\} \), we may denote this intersection by the index sets \( \mathbb{O} = \{1, 4\} \) and \( \mathbb{I} = \{2\} \). We note that if \( \mathbb{O} \cap \mathbb{I} \neq \emptyset \), there is one index \( i \in \mathbb{O} \cap \mathbb{I} \) for which our collection of hyperplanes to be intersected includes both \( \{x \in \mathbb{R}^n : x_1 = 0\} \) and \( \{X \in \mathbb{R}^n : x_1 = 1\} \) and the intersection is consequently empty.

Let us now have a small look at the vertex sets generated by the hyperplane intersection corresponding to a pair of in-set and out-set.

Observation 25. Let \( \mathbb{O}, \mathbb{I} \subseteq [n] \). Then:

\[
\{X \in C(n; a) : \forall i \in \mathbb{I} : i \in X, \forall j \in \mathbb{O} : j \not\in X\} = \{X \in C(n; a) : \mathbb{I} \subseteq X \subseteq [n] \setminus \mathbb{O}\}
\]

This equality is due to the equivalence between the statements \( \forall i \in \mathbb{I} : i \in X \) « expressed in terms of single objects \( i \in [n] \) and \( \forall \mathbb{I} \subseteq X \), expressed in terms of the entire set at once.

Let us continue by examining the min and max hyperplane bounding the polytope \( P(n; a) \) from both sides along the direction of the all-ones vector 1.
3.2 Vertices on min/max-Hyperplanes

For the min/max-hyperplanes, we defined $V_B$ to represent the vertices complying with the hyperplane of the cardinality bounds $a_1$ and $a_m$, respectively.

**Proposition 26.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. Then:

$$V_B(n; a; \text{min}) = \binom{[n]}{a_1} \quad \text{and} \quad V_B(n; a; \text{max}) = \binom{[n]}{a_m}.$$  

**Proof.** For the min-hyperplane, we may reformulate $V_B(n; a; \text{min}) := \{X \in C(n; a) : a_1 = \sum_{i=1}^n \chi_X(i)\} = \{X \in C(n; a) : a_1 = |X|\} = \{X \in \bigcup_{j=1}^m \binom{[n]}{a_j} : a_1 = |X|\} = \binom{[n]}{a_1}$.

Analogously, for the max-hyperplane, $V_B(n; a; \text{max}) = \{X \in C(n; a) : a_m = \sum_{i=1}^n \chi_X(i)\} = \{X \in C(n; a) : a_m = |X|\} = \{X \in \bigcup_{j=1}^m \binom{[n]}{a_j} : a_m = |X|\} = \binom{[n]}{a_m}$.

The behaviour of min- and max-hyperplane is plain and simple: It contains all the sets of the smallest (min-hyperplane)/largest (max-hyperplane) cardinality listed in $a$. We shall use a **bound-set** $\mathbb{M} \subseteq \{\min, \max\}$ to denote if none, one or both of the min/max-hyperplanes are to be intersected with. Figure 3 illustrates the vertex sets on the hyperplanes for $C(5; (1, 4))$.

Figure 3: Vertices of $C(5; (1, 4))$ on the min/max-hyperplane in black

Clearly, the intersection of $V_B(n; a; \text{min})$ and $V_B(n; a; \text{max})$ is empty if and only if $a_1 \neq a_m$, which is the case if and only if $m > 1$. In the case $m = 1$, the entire polytope $P(n; a)$ is contained in both of the two hyperplanes, as then

$$C(n; a) = \bigcup_{i \in [m]} \binom{[n]}{a_i} = \binom{[n]}{a_1} = \binom{[n]}{a_m}.$$  

For a CF-hyperplane, it is not so clear which sets lie on it, nor is it clear what their intersections look like. We will study this topic next.

3.3 Vertices on CF-Hyperplanes

We recall that CF-hyperplanes exist only for sets $S \in \mathcal{S}(n; a) = \{S \subseteq [n] : a_1 < |S| < a_m, |S| \notin \{a_1, \ldots, a_m\}\}$. If $|S| = a_1$ were the case for some layer $\ell \in [m]$, its incidence vector would be contained in $P(n; a)$, even be a vertex of the polytope, and not be someting to be cut off from the
n-cube, which was a way to understand CF-inequalities in the first place. Sets $S$ with $|S| < a_1$ or $a_m < |S|$ are not yielding CF-inequalities either as the polytope is restricted to the area in between by the top and bottom inequalities anyway and nothing beyond these bounds has to be cut off. So each of the sets $S \in S(n; a)$ lies in some gap $g \in [m - 1]$ and has a cardinality $|S|$ in between $a_g$ and $a_{g+1}$. We will now see that any CF-hyperplane associated with a set $S$ contains only vertices which lie on one of the two adjacent layers.

**Proposition 27.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. Let $S \in S(n; a)$ be a set in gap $g \in [m - 1]$. Then:

$$V_{CF}(n; a; S) = \{ X \in \binom{[n]}{a_g} : X \subset S \} \cup \{ X \in \binom{[n]}{a_{g+1}} : S \subset X \}.$$  

**Proof.** We recall the equality a set $X \in C(n; a)$ has to satisfy in order to be contained in $V_{CF}(n; a; S)$ and find some more handy reformulations, exploiting that $|A| = |A \cap B| + |A \setminus B|$.

$$\sum_{i \in S}(a_{g+1} - |S|)|X(i) - \sum_{i \in [n]\not\in S}(|S| - a_g)|X(i) = (a_{g+1} - |S|)a_g \iff (a_g - |S|)1^T X + (a_{g+1} - a_g)X^T_S X = (a_{g+1} - |S|)a_g \iff (a_g - |S|)|X| + (a_{g+1} - a_g)|X \cap S| = (a_{g+1} - |S|)a_g \iff a_g|X| + a_g|S| + a_{g+1}|X \cap S| - a_g|X \cap S| = |S||X| + a_ga_{g+1} \iff |X \setminus S|(a_g - a_{g+1}) = (a_{g+1} - |S|)(a_g - |X|) \tag{28} \iff |S \setminus X|(a_g - a_{g+1}) = (|X| - a_{g+1})(|S| - a_g) \tag{29}$$

Let us now examine which sets $X \in C(n; a)$ satisfy these equivalent equalities $28$ and $29$. We recall that $a_g < |S| < a_{g+1}$ and distinguish cases by the cardinality of $X$:

- **If** $|X| \leq a_g$, **we try to apply Equality** $28$ and observe that $|X \setminus S| \geq 0$ and $a_g - a_{g+1} < 0$, so the left-hand side is non-positive. On the right-hand side, $a_{g+1} - |S| > 0$ always holds.
  - If $|X| < a_g$, then $a_g - |X| > 0$, so the right-hand side is positive, therefore no $X \in C(n; a)$ with $|X| < a_g$ satisfies $28$.
  - If $|X| = a_g$, then $a_g - |X| = 0$, so the right-hand side is zero. Equality holds if and only if the left-hand side is zero as well, which is the case whenever $|X \setminus S| = 0$, which, as $|X| < |S|$, is equivalent to $X \subset S$.

- **If** $|X| \geq a_{g+1}$, **we try to apply Equality** $29$ and observe that $|S \setminus X| \geq 0$ and $a_g - a_{g+1} < 0$, so the left-hand-side is non-positive. On the right-hand side, $|S| - a_g > 0$ always holds.
  - If $|X| = a_{g+1}$, then $|X| - a_{g+1} = 0$, so the right-hand side is zero. Equality holds if and only if the left-hand side is zero as well, which is the case whenever $|S \setminus X| = 0$, which, as $|S| < |X|$, is equivalent to $S \subset X$.
  - If $|X| > a_{g+1}$, then $|X| - a_{g+1} > 0$, so the right-hand side is positive, therefore no $X \in C(n; a)$ with $|X| > a_{g+1}$ satisfies $29$.

We may now directly conclude the desired statement. 

Let us take a look at some vertex sets on CF-hyperplanes for our examplary $C(5; (1, 4))$. We first observe that $S(5; (1, 4)) = \{ S \subset [5] : 1 < |S| < 4, |S| \not\in \{1, 4\} \} = \binom{\{2\}}{2} \cup \binom{\{3\}}{3}$. We look
at one example of a set of cardinality 2 and one for a set of cardinality 3 in Figure 4. In both diagrams, the set $S$ is indicated in red.

Let us now find out what the vertex set of an intersection of CF-hyperplanes, that is, the intersection of the vertex sets of some CF-hyperplanes, looks like. In line with the in-set $I$ and out-set $O$ used for $0/1$-hyperplanes and the bound-set $M$ for min/max-hyperplanes, we define the collection $S \subseteq S(n; a)$ to denote a CF-family of such sets. Subsequently, we will introduce a classification of CF-families $S$ that will allow us to determine $\bigcap_{S \in S} V_{\text{CF}}(n; a; S)$ more explicitly for each of the classes.

**Definition 30 (CF-family).** Let $S \subseteq S(n; a)$ be a collection of sets $S$ with $S \in S(n; a)$. $S$ shall be called a CF-family. We can partition CF-families into the following four classes:

- **$S = \emptyset$** is the empty CF-family.
- Let $S \subseteq S(n; a)$ be a non-empty CF-family with the property that all $S \in S$ are in the same gap $g$. Such an $S$ shall be called a CF-family in gap $g$. We define
  
  $S^U := \bigcup_{S \in S} S = \{ i \in [n] : \exists S \in S : i \in S \}$ and
  
  $S^C := \bigcap_{S \in S} S = \{ i \in [n] : \forall S \in S : i \in S \}$

- Let $S_{\ell - 1} \subset S(n; a)$ be a CF-family in gap $(\ell - 1)$ and $S_\ell \subset S(n; a)$ be a CF-family in gap $\ell$ for some $\ell \in \{2, \ldots, m - 1 \}$. Their union $S := S_{\ell - 1} \cup S_\ell$ shall be called a CF-family around layer $\ell$.

- Let $S \subseteq S(n; a)$ be a non-empty CF-family that is neither a CF-family in gap $g$ for any $g \in [m - 1]$ nor a CF-family around layer $\ell$ for any $\ell \in \{2, \ldots, m - 1 \}$. Such a family $S$ shall be called a wild CF-family.

We note that the non-emptiness of a CF-family in a gap gives us the certainty that any CF-family around a layer does contain at least one set out of each of the two adjacent gaps, ensuring that no CF-family can be a CF-family in a gap and a CF-family around a layer at the same time. As mentioned before, this classification now enables us to determine the set of vertices of a CF-family. We will state and prove descriptions of the vertex set for the classes of non-empty CF-families.
Proposition 31. Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. Let $S \subseteq S(n; a)$ be a non-empty CF-family. Depending on its class, we may state the following:

1. If $S$ is a CF-family in gap $g$ with $g \in [m - 1]$, then:

$$\bigcap_{S \in S} V_{CF}(n; a; S) = \{X \in \binom{[n]}{a_g} : X \subseteq S \} \cup \{X \in \binom{[n]}{a_{g+1}} : X \supseteq S^\cup \}$$

2. If $S = S_{\ell-1} \cup S_{\ell}$ is a CF-family around layer $\ell$ with $S_{\ell-1}$ a CF-family in gap $(\ell - 1)$ and $S_{\ell}$ a CF-family in gap $\ell$, then:

$$\bigcap_{S \in S} V_{CF}(n; a; S) = \{X \in \binom{[n]}{a_{\ell}} : S_{\ell-1} \subseteq X \subseteq S_{\ell} \}$$

3. If $S$ is a wild CF-family, then:

$$\bigcap_{S \in S} V_{CF}(n; a; S) = \emptyset$$

Proof. We will prove the three statements one after another.

1. We plug in Definition [27] then we exploit the fact that $\binom{[n]}{a_g}$ and $\binom{[n]}{a_{g+1}}$ are disjoint and apply the definitions of $S^\cap$ and $S^\cup$.

$$\bigcap_{S \in S} V_{CF}(n; a; S) = \bigcap_{S \in S} \left(\{X \in \binom{[n]}{a_g} : X \subset S\} \cup \{X \in \binom{[n]}{a_{g+1}} : X \supset S\}\right)$$

$$= \bigcap_{S \in S} \{X \in \binom{[n]}{a_g} : X \subset S\} \cup \bigcap_{S \in S} \{X \in \binom{[n]}{a_{g+1}} : X \supset S\}$$

$$= \{X \in \binom{[n]}{a_g} : \forall S \in S : X \subset S\} \cup \{X \in \binom{[n]}{a_{g+1}} : \forall S \in S : X \supset S\}$$

$$= \{X \in \binom{[n]}{a_g} : X \subseteq S^\cap\} \cup \{X \in \binom{[n]}{a_{g+1}} : X \supseteq S^\cup\}$$

In the last step, we used that $\forall S \in S : X \subset S$ is equivalent to $\forall S \in S : X \subseteq S^\cap$. The reason for which the latter does imply the former with strict inclusion is that we only have sets $X$ of cardinality $a_g$ and all the sets $S \in S$ have a cardinality strictly larger than $a_g$. The equivalence for the $X$ of cardinality $a_{g+1}$ can be reasoned analogously.

2. We split $S$ back into $S_{\ell-1} \cup S_{\ell}$ and plug in what we learned about the vertices of a CF-family in a gap. We then observe that the intersection contains sets from $\binom{[n]}{a_{\ell}}$ only and simplify the statement to get the desired equality.

$$\bigcap_{S \in S} V_{CF}(n; a; S) = \bigcap_{S \in S_{\ell-1}} V_{CF}(n; a; S) \cap \bigcap_{S \in S_{\ell}} V_{CF}(n; a; S)$$

$$= \left(\{X \in \binom{[n]}{a_{\ell-1}} : X \subseteq S_{\ell-1}\} \cup \{X \in \binom{[n]}{a_{\ell}} : X \supseteq S_{\ell-1}\}\right)$$

$$\cap \left(\{X \in \binom{[n]}{a_{\ell}} : X \subseteq S_{\ell}\} \cup \{X \in \binom{[n]}{a_{\ell+1}} : X \supseteq S_{\ell}\}\right)$$
\[\{X \in \binom{[n]}{a_\ell} : X \supset S_{\ell-1}^{(\ell)} \}\cap \{X \in \binom{[n]}{a_\ell} : X \subset S_{\ell}^{(\ell)}\}\]

\[\{X \in \binom{[n]}{a_\ell} : S_{\ell-1}^{(\ell)} \subset X \subset S_{\ell}^{(\ell)}\}\]

3. Let \(S \subseteq S(n;a)\) be a wild CF-family. For each gap \(g \in [m-1]\), we introduce \(S_g := \{S \subseteq S : a_g < |S| < a_{g+1}\}\), so that \(S = \bigcup_{g \in [m-1]} S_g\). Let \(K = \{i \in [m-1] : S_i \neq \emptyset\}\) be the collection of the indices of those gaps for which there is at least one set in \(S\). Consequently, \(S = \bigcup_{i \in K} S_i\).

- If \(k \leq 1\), \(S\) is either a CF-family in a gap or an empty CF-family, but not a wild one.
- If \(k = 2\) and \(K = \{j, j+1\}\) for some \(j \in [m-2]\), \(S\) is a CF-family around layer \(j+1\), in contradiction to the assumption that \(S\) is a wild CF-family.
- If \(k \geq 2\) and \(\not \exists j \in [m-2] : K = \{j, j+1\}\), there are \(x, y \in K, x + 1 < y\). Then \(S_x \cup S_y \subseteq S\) and consequently,

\[\bigcap_{S \in S} V_{CF}(n;a;S) \subseteq \bigcap_{S \in S_x} V_{CF}(n;a;S) \cap \bigcap_{S \in S_y} V_{CF}(n;a;S) \subseteq (\binom{[n]}{a_x} \cup \binom{[n]}{a_x+1}) \cap (\binom{[n]}{a_y} \cup \binom{[n]}{a_y+1}) = \emptyset,\]

as we saw that any CF-family in a gap only contains sets on the adjacent layers and \(x+1 < y\) implies that they do not have any adjacent layer in common.

\[\Box\]

Let us use this result to state an immediate consequence: We acquired some knowledge as to when vertices of a certain cardinality can be contained in the vertex set of the intersection of some CF-hyperplanes.

**Observation 32.** Let \(S\) be a non-empty CF-family and \(j \in [m]\) indicate a cardinality \(a_j\) in the cardinality sequence. Proposition 31 directly implies that \(\bigcap_{S \in S} V_{CF}(n;a;S) \cap \binom{[n]}{a_j}\) can only be non-empty if \(S\) is a CF-family in gap \(j-1\) or \(j\) or a CF-family around layer \(j\).

4 A Classification of the Faces

So far, we have translated the geometrical problem of finding the faces of \(P(n;a)\) into the combinatorial problem of intersecting subsets of \(C(n;a)\) for which the set of associated incidence vectors is the vertex set of one of the polytope’s faces.

Let us introduce a universal notation to denote the vertex set constructed by intersecting \(C(n;a)\) with some of the hyperplanes of the three classes established in the previous section.

**Definition 33 (V\((n;a;M;\mathbb{I};\emptyset;S)\)).** Let \(n \in \mathbb{N}\) and \(a\) be a cardinality sequence. For \(\mathbb{I}, \emptyset \subseteq [n]\), \(\mathbb{M} \subseteq \{\text{min}, \text{max}\}\) and \(S \subseteq S(n;a)\), we define

\[V(n;a;M;\mathbb{I};\emptyset;S) := \left\{X \in C(n;a) : \begin{array}{l}
\forall M \in \mathbb{M} : X \in V_B(n;a;M) \text{ and } \\
\forall i \in \mathbb{I} : X \in V_1(n;a;i) \text{ and } \\
\forall j \in \emptyset : X \in V_0(n;a;j) \text{ and } \\
\forall S \in S : X \in V_{CF}(n;a;S) \end{array}\right\} \]
We observe the following:

- If we set $\mathcal{M} = I = O = S = \emptyset$, then $V(n;a;\emptyset;\emptyset;\emptyset) = C(n;a)$ is the vertex set of the polytope $P(n;a)$ itself, intersected with none of the hyperplanes.

- Due to the way that $\forall \in \mathbb{R}$ appears, we note $V(n;a;\mathcal{M};I;O;S) = V(n;a;\mathcal{M};\emptyset;\emptyset;\emptyset) \cap V(n,a;\emptyset;\emptyset;\emptyset) \cap V(n,a;\emptyset;\emptyset;\emptyset;S) = V(n,a;\mathcal{M};I;O;S_1 \cup S_2) \cap V(n,a;\mathcal{M};I;O;S) = \{X \in \cap_{S \in S} V_{C_{F}}(n;a;S) : I \subseteq X \subseteq [n] \setminus O\}$. Let us first determine the dimension of the polytope $P(n;a)$ itself, intersected with none of the hyperplanes.

We note that in this definition for $V(n;a;\mathcal{M};I;O;S)$, each class of hyperplanes, be it min-, max-, 1-, 0- or $C_{F}$-hyperplanes, is condensed into sets $\mathcal{M}, S, I$ and $O$, respectively, which we defined in the previous section. Let us make some more observations:

**Observation 34.** We observe the following:

- If we set $\mathcal{M} = I = O = S = \emptyset$, then $V(n;a;\emptyset;\emptyset;\emptyset) = C(n;a)$ is the vertex set of the polytope $P(n;a)$ itself, intersected with none of the hyperplanes.

- Due to the way that $\forall \in \mathbb{R}$ appears, we note $V(n;a;\mathcal{M};I;O;S) = V(n;a;\mathcal{M};\emptyset;\emptyset;\emptyset) \cap V(n,a;\emptyset;\emptyset;\emptyset) \cap V(n,a;\emptyset;\emptyset;\emptyset;S) = V(n,a;\mathcal{M};I;O;S_1 \cup S_2) \cap V(n,a;\mathcal{M};I;O;S) = \{X \in \cap_{S \in S} V_{C_{F}}(n;a;S) : I \subseteq X \subseteq [n] \setminus O\}$. Let us first determine the dimension of the polytope $P(n;a)$ itself, intersected with none of the hyperplanes.

These sets $V(n;a;\mathcal{M};I;O;S)$ will be the objects we are dealing with from now on. Let us introduce some more definitions to be able to handle them properly.

**Definition 35 (Properties of the Faces).** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. For $I, O \subseteq [n], \mathcal{M} \subseteq \{\min, \max\}$ and $S \subseteq S(n;a), F$ shall give the face associated with a vertex set, that is, the convex hull of the vertices. $d$ shall indicate the dimension of that face, while $L$ shall be the set of those layers which the vertex set intersects:

\[
F(n;a;\mathcal{M};I;O;S) := \text{conv}\{X \in \{0,1\}^n : X \in V(n;a;\mathcal{M};I;O;S)\}
\]

\[
d(n;a;\mathcal{M};I;O;S) := \dim(F(n;a;\mathcal{M};I;O;S))
\]

\[
L(n;a;\mathcal{M};I;O;S) := \{k \in [m] : \binom{n}{a_k} \cap V(n;a;\mathcal{M};I;O;S) \neq \emptyset\}.
\]

We will now appropriate these definitions to describe the faces built by vertex sets of the form $V(n;a;\mathcal{M};I;O;S)$. Let us first determine the dimension of the polytope $P(n;a)$ itself. As its vertex set is equal to $V(n;a;\emptyset;\emptyset;\emptyset;\emptyset)$, its dimension is denoted by $d(n,a;\emptyset;\emptyset;\emptyset;\emptyset)$. The dimension of the polytope itself has already been worked out by Grötschel (2004):

**Proposition 36.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. The dimension of $P(n;a)$ is the following:

\[
d(n,a;\emptyset;\emptyset;\emptyset;\emptyset) = \begin{cases} 
0, & m = 1, a_1 \in \{0, n\} \\
n - 1, & m = 1, a_1 \in \{1, \ldots, n - 1\} \\
1, & m = 2, a = (0, n) \\
n, & \text{else} 
\end{cases}
\]

Let us take a moment to realise where we are standing right now. The reason we constructed all these tools is, as we argued earlier in a less formal manner, that what we get through them are exactly the vertex sets of faces of $P(n;a)$. Let us state and prove this now.
Proposition 37. Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence for $n$. Then the faces of $P(n; a)$ are exactly the sets $F(n; a; \mathbb{M}; \mathbb{I}; \mathbb{O}; \mathbb{S})$.

Proof. We start by reformulating the statement more formally as

$$\{ F \subseteq \mathbb{R}^n : F \text{ is a face of } P(n; a) \} = \left\{ F(n; a; \mathbb{M}; \mathbb{I}; \mathbb{O}; \mathbb{S}) : \mathbb{M} \subseteq \{ \text{min, max} \} \text{ and } \mathbb{S} \subseteq \mathbb{S}(n; a) \right\}.$$

We recall that Proposition 21 stated that for every face $F$ there is a set of hyperplanes $H \subseteq \mathcal{H}(n; a)$ such that $F$ is equal to the intersection of $P(n; a)$ with the hyperplanes in $H$. Conversely, Proposition 20 stated that any intersection of $P(n; a)$ with some hyperplanes collected in a set $H \subseteq \mathcal{H}(n; a)$ is a face of $P(n; a)$. We conclude as a first step that

$$\{ F \subseteq \mathbb{R}^n : F \text{ is a face of } P(n; a) \} = \{ P(n; a) \cap \bigcap_{h \in H} h : H \subseteq \mathcal{H}(n; a) \}.$$

Let us use the statement of Proposition 11.3 revealing that any face $F$ is a polytope with $\text{vert}(F) = F \cap \text{vert}(P)$ and that of Proposition 11.1 implying that $F$ is the convex hull of its vertices. We can combine this into the equality $F = \text{conv}(F \cap \text{vert}(P(n; a)))$ for every face $F$ of $P(n; a)$ and apply this equality for the faces we found to have the form $P(n; a) \cap \bigcap_{h \in H} h$:

$$\{ P(n; a) \cap \bigcap_{h \in H} h : H \subseteq \mathcal{H}(n; a) \} = \{ \text{conv}(\bigcap_{h \in H} (\text{vert}(P(n; a)) \cap h)) : H \subseteq \mathcal{H}(n; a) \}.$$

It is now left to show that the sets $\bigcap_{h \in H} (\text{vert}(P(n; a)) \cap h)$ for a collection $H$ of hyperplanes in $\mathcal{H}(n; a)$ are exactly the sets of incidence vectors of $V(n; a; \mathbb{M}; \mathbb{I}; \mathbb{O}; \mathbb{S})$ for feasible choices of input variables. The way we defined the sets $\mathbb{M}$, $\mathbb{I}$, $\mathbb{O}$ and $\mathbb{S}$, each of them indicates some hyperplanes in $\mathcal{H}(n; a)$, for one of their classes $(0, 1, \text{min/max}, \text{CF})$. The way we defined $V(n; a; \mathbb{M}; \mathbb{I}; \mathbb{O}; \mathbb{S})$, it contains exactly those objects of $C(n; a)$ whose incidence vectors lie on each hyperplane collected in the sets $\mathbb{M}$, $\mathbb{I}$, $\mathbb{O}$ and $\mathbb{S}$, which is exactly what $\bigcap_{h \in H} (\text{vert}(P(n; a)) \cap h)$ does when collecting the same hyperplanes by their row indices in the matrix $A(n; a)$ we defined earlier. Therefore, these are just two ways to represent any such set on either side. As identical sets have identical convex hulls, we conclude that the faces of $P(n; a)$ are indeed exactly the sets $F(n; a; \mathbb{M}; \mathbb{I}; \mathbb{O}; \mathbb{S})$.

We deduce from this statement that in classifying, counting and describing all the point sets $F(n; a; \mathbb{M}; \mathbb{I}; \mathbb{O}; \mathbb{S})$, we classify, count and describe exactly the faces of $P(n; a)$.

4.1 Descriptions of Faces

For a concise description of the faces, we will introduce some more notions related to polytopes. Again, we follow the definitions of [Ziegler, 1995].

Definition 38 (Isomorphisms of Polytopes). Two polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ are **affinely isomorphic** if there is an affine map $f : \mathbb{R}^d \to \mathbb{R}^e$ that is a bijection between the points of the two polytopes. $P$ and $Q$ are **combinatorially equivalent** (denoted by $P \simeq Q$), if there is a bijection between their faces that preserves the inclusion relation. The equivalence classes of $\simeq$ are called the **combinatorial types** of polytopes.
**Definition 39** (Simplex, Pyramid and Join). The **standard d-simplex** $\Delta_d$ is the convex hull of the unit vectors in $\mathbb{R}^{d+1}$. For a polytope $P$ and a point $x$ outside the affine hull of $P$, the **pyramid** over $P$ is the convex hull $\text{pyr}(P) := \text{conv}(P \cup \{x\})$. The **join** $P \ast Q$ of two polytopes is the convex hull of $P$ and $Q$, if they are placed into affine subspaces of some $\mathbb{R}^d$ such that their affine hulls $\text{aff}(P)$ and $\text{aff}(Q)$ are skew.

We note that any two affinely isomorphic polytopes are combinatorially equivalent and all polytopes of the same combinatorial type have the same dimension. As for the standard-$d$-simplex, we observe that the unit vectors in $\mathbb{R}^{d+1}$ are the incidence vectors of $C(d+1; (1))$, so $\Delta_d$ is in fact the polytope of a cardinality homogeneous set system, $\Delta_d = P(d+1; (1))$. Furthermore, we observe that any convex hull of two distinct vectors is combinatorially equivalent to both the 1-simplex $\Delta_1$ and a pyramid over a single point. Ziegler (1995) notes some more facts about the special polytopes we just defined: The dimension of $\Delta_d$ is $d$ and the dimension of $\text{pyr}(P)$ is $\dim(P) + 1$, while $\dim(P \ast Q) = \dim(P) + \dim(Q) + 1$. Moreover, the combinatorial type of a pyramid is independent of the choice of $x$, and the combinatorial type of the join is independent of the choice of affine subspaces.

With the help of these definitions, we are now prepared to determine for each face of $P(n; a)$ not only the dimension, but also the combinatorial type by showing combinatorial equivalence to a lower-dimensional polytope of a cardinality homogeneous set system or a polytope constructed as the pyramid of one or the join of two such polytopes.

### 4.2 Some Configurations to Be Discarded

In order to examine the faces, we will introduce a classification by which we may partition the plethora of vertex sets of the form $V(n; a; M; I; O; S)$ into disjoint classes. These classes will be characterised by the number of different layers a vertex set spans and the distribution of the vertex set among those layers. We will elaborate the scheme later on.

At this point, let us first examine the behaviour of certain configurations $(n; a; M; I; O; S)$ where the sets $M$ and $S$ are empty, non-empty or have a certain property. After discussing these configurations, we will be able to narrow down our range of hyperplane combinations to be taken into account considerably. Table 1 shows all combinations of properties for $M$ and $S$.

<table>
<thead>
<tr>
<th>$M = {\text{min, max}}$</th>
<th>$S$ empty</th>
<th>$S$ in a gap</th>
<th>$S$ around a layer</th>
<th>$S$ wild</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>discarded</strong> (Prop. 42)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>analysed</strong> in Subsection 4.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M = {\text{min}}$ or $M = {\text{max}}$</td>
<td></td>
<td></td>
<td><strong>discarded</strong> (Prop. 41)</td>
<td></td>
</tr>
<tr>
<td><strong>analysed</strong> in Subsection 4.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M = \emptyset$</td>
<td></td>
<td></td>
<td><strong>analysed</strong> in Subsection 4.6</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The configurations of vertex sets

We note that there is nothing to find out about faces of dimension $-1$ or $0$. The only face of dimension $-1$ is known to be the empty set for every polytope, while the sets of dimension $0$ we determined to be exactly the incidence vectors of sets in $C(n; a)$. We will therefore discard any configuration that never yields any set $V(n; a; M; I; O; S)$ containing at least two elements. Let us point out a configuration to violate this condition. Proposition 31 tells us that a vertex set...
that lies on every hyperplane of a wild CF-family is necessarily empty. The following is a direct consequence of this fact and causes us to discard any configuration where $S$ is wild.

**Proposition 40.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $I, O \subseteq [n]$ and $M \subseteq \{ \min, \max \}$. If $S \subseteq S(n; a)$ is a wild CF-family, then $V(n; a; M; I; O; S)$ is empty.

Furthermore, there are some configurations for which any set generated by such a configuration can also be generated by another non-discarded configuration. We will discard those configurations as well to narrow down our analysis to as few configurations as possible. An example for this are the vertex sets where $S$ is a CF-family around a layer. Any such vertex set can equally be represented with slightly altered in- and out-sets and an empty CF-family.

**Proposition 41.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $I, O \subseteq [n]$ and $M \subseteq \{ \min, \max \}$. If $S \subseteq S(n; a)$ is a CF-family around layer $\ell$ with $S = S_{\ell-1} \cup S_{\ell}$, where $S_{\ell-1}$ is a CF-family in gap $\ell-1$ and $S_{\ell}$ is a CF-family in gap $\ell$, then $V(n; a; M; I; O; S) = V(n; a; M; I \cup S_{\ell-1}; O \cup [n] \setminus S_{\ell}; \emptyset)$.

**Proof.** We start by recalling that we may split $V(n; a; M; I; O; S)$ into $V(n; a; M; I; O; S) = V(n; a; M; I; O; S) \cap V(n; a; \emptyset; \emptyset; \emptyset; S)$. As $S$ is a CF-family around layer $\ell$, Proposition 31 confirms the last equality in $V(n; a; \emptyset; \emptyset; \emptyset; S) = \{ X \in C(n; a) : \forall S \in S : X \in V_{\text{CF}}(n; a; S) \} = \{ X \in C(n; a) : X \in \bigcap_{S \in S} V_{\text{CF}}(n; a; S) \}$.

At this point, we may exploit that $\forall S \in S : |S| < a_{\ell+1}$ implies $|S_{\ell}| < a_{\ell+1}$, so no $X$ of cardinality $\leq a_{\ell+1}$ can be a subset of $S_{\ell}$. On the other hand, $\forall S \in S : a_{\ell-1} < |S|$ implies $a_{\ell-1} < |S_{\ell-1}|$, so analogously, no $X$ of cardinality $\leq a_{\ell-1}$ can be a superset of $S_{\ell-1}$.

Therefore, $\{ X \in \binom{[n]}{a} : S_{\ell-1} \subseteq X \subseteq S_{\ell} \} = \{ X \in C(n; a) : S_{\ell-1} \subseteq X \subseteq S_{\ell} \} = V(n; a; \emptyset; S_{\ell-1}; [n] \setminus S_{\ell}; \emptyset)$ and we conclude that $V(n; a; M; I; O; S) = V(n; a; M; I; O; \emptyset) \cap V(n; a; \emptyset; \emptyset; S) = V(n; a; M; I; O; \emptyset) \cap V(n; a; \emptyset; S_{\ell-1}; [n] \setminus S_{\ell}; \emptyset) = V(n; a; M; I \cup S_{\ell-1}; O \cup [n] \setminus S_{\ell}; \emptyset)$ where the last equality is once more due to the $\forall$-style in Definition 33.

The next configuration we will discard are the vertex sets where $M$ indicates to intersect with both the min- and the max-hyperplane. We will show that any resulting vertex set is empty or equal to a similar configuration with $M = \emptyset$.

**Proposition 42.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $I, O \subseteq [n]$ and $S \subseteq S(n; a)$. If $M = \{ \min, \max \}$, then either $V(n; a; M; I; O; S) = \emptyset$ or $V(n; a; M; I; O; S) = V(n; a; M; I; O; \emptyset)$.

**Proof.** Let us start by splitting $V(n; a; M; I; O; S) = V(n; a; M; \emptyset; \emptyset; \emptyset) \cap V(n; a; \emptyset; I; O; S)$. We now observe that $V(n; a; M; \emptyset; \emptyset; \emptyset) = \{ X \in C(n; a) : X \in V_{\max}(n; a; \min) \cap V_{\max}(n; a; \max) \} = \{ X \in C(n; a) : X \in \binom{[n]}{a} \cap \binom{[n]}{a} \}$.

We note that for $m = 1$, $a_1 = a_m$, so this set is equal to $C(n; a)$, thus $V(n; a; M; I; O; S) = C(n; a) \cap V(n; a; \emptyset; I; O; S) = V(n; a; \emptyset; I; O; S)$. For $m > 1$, $a_1 \neq a_m$ and therefore $\{ X \in C(n; a) : X \in \binom{[n]}{a_1} \cap \binom{[n]}{a_m} \}$ is empty. Then $V(n; a; M; I; O; S) = \emptyset$ and $V(n; a; \emptyset; I; O; S) = \emptyset$.

One last configuration we wish to discard are those vertex sets where $M$ contains exactly one of the min- and max-hyperplanes and $S$ is a CF-family in a gap. Such vertex sets are either empty or equally representable with an empty CF-family.
Proposition 43. Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $\mathcal{I}, \mathcal{O} \subseteq [n]$ and $\mathcal{S} \subseteq S(n; a)$ be a CF-family in gap $g$. If $\mathcal{M} = \{\text{min}\}$ or $\mathcal{M} = \{\text{max}\}$, then

$$V(n; a; \mathcal{M}; \mathcal{I} \cup \mathcal{O} \cup S) = \begin{cases} V(n; a; \mathcal{M}; \mathcal{I}; \mathcal{O} \cup [n] \setminus \mathcal{S} \cap \emptyset) & \text{if } \mathcal{M} = \{\text{min}\}, g = 1 \\ V(n; a; \mathcal{M}; \mathcal{I} \cup S \cup \mathcal{O}; \emptyset) & \text{if } \mathcal{M} = \{\text{max}\}, g = m - 1 \\ \emptyset & \text{in all other cases.} \end{cases}$$

Proof. Let us start by converting the set we examine into an intersection of three other sets: $V(n; a; \mathcal{M}; \mathcal{I} \cup \mathcal{O} \cup S) = V(n; a; \mathcal{M}; \emptyset; \emptyset; \emptyset) \cap V(n; a; \emptyset; \emptyset; \mathcal{S}) \cap V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{O})$. We see that $V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{S}) = \{X \in C(n; a) : X \in \bigcap_{S \in \mathcal{S}} V_{CF}(n; a; S)\}$. Let $X \in \binom{[n]}{a}$ : $X \subseteq \mathcal{S} \cap \emptyset \cup \{X \in \binom{[n]}{a} : X \subseteq \mathcal{S} \cup \mathcal{O} \cup \emptyset\}$. We see that $X \in \bigcap_{S \in \mathcal{S}} V_{CF}(n; a; S)$ and consequently, the intersection with $V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{O})$ is non-empty if and only if $g = 1$, in which case $V(n; a; \mathcal{M}; \emptyset; \emptyset; \emptyset) \cap V(n; a; \emptyset; \emptyset; \mathcal{S}) \cap V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{O}) = V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{O})$. If $\mathcal{M} = \{\text{max}\}$, then $V(n; a; \mathcal{M}; \emptyset; \emptyset; \emptyset) = \binom{[n]}{m}$ and consequently, the intersection with $V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{S})$ is non-empty if and only if $g = m - 1$, in which case $V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{O}) \cap V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{S}) = V(n; a; \emptyset; \emptyset; \emptyset; \mathcal{O})$.

Although we did not discard any configuration with certain in- and out-sets $\mathcal{I}$ and $\mathcal{O}$ here, let us recall that $\mathcal{I} \cap \mathcal{O} \neq \emptyset$ causes the associated vertex set to be empty. We shall therefore request $\mathcal{I} \cap \mathcal{O} = \emptyset$ in all of the configurations we analyse.

4.3 A Classification Scheme

After having narrowed down the configurations we need to consider, let us now introduce a classification of the vertex sets, grouping vertex sets with certain properties to be able to describe and count them precisely. We recall that we are not interested in empty or one-element vertex sets and shall therefore only classify vertex sets containing at least two elements:

Definition 44 (Classes of Vertex Sets). Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $V \subseteq C(n; a)$ be a non-empty vertex set containing at least two elements.

If all the vertices $X \in V$ are on the same layer $\ell \in [m]$, then $V$ shall be called a full-$1$-layer set. The vertex sets containing vertices on exactly two different layers $\ell < \ell'$ shall be named depending on the distribution of the vertices among the layers: If $V$ contains exactly one vertex on each of the two layers $\ell$ and $\ell'$, we shall call it a 1-$1$-pyramid set. If it contains exactly one vertex on layer $\ell$ and at least two vertices on layer $\ell'$, we shall call it a 1-many-pyramid set. If it contains at least two vertices on layer $\ell$ and exactly one vertex on layer $\ell'$, we shall call it a many-$1$-pyramid set. If it contains at least two vertices on each of the two layers $\ell$ and $\ell'$, we shall call it a full-$2$-layer set. A vertex set containing vertices on at least three different layers shall be called a many-layer set.

Clearly, these definitions cover all cases for vertex sets of cardinality at least two. As any vertex in $C(n; a)$ is uniquely associated with a layer, any two of the classes in boldface are disjoint and together, they partition the set of all vertex sets of the form $V(n; a; \mathcal{M}; \mathcal{I} \cup \mathcal{O} \cup S)$.

We will now go through those three configurations in Table 1 which we did not discard and analyse their vertex sets using the above classification.
4.4 Vertex Sets for $\mathbb{M} = \emptyset$ and $S = \emptyset$

**Proposition 45.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $\emptyset, O \subseteq [n]$ with $\emptyset \cap O = \emptyset$. Then:

$$V(n; a; \emptyset; O; \emptyset) = \bigcup_{k=1}^{m} \{ \emptyset \cup Y : Y \in \binom{[n] \setminus (\emptyset \cup O)}{a_k - |\emptyset|} \}$$

**Proof.** We first plug in the definitions of $V_0(n; a; i)$ and $V_1(n; a; j)$ all at once. Then we split the set by cardinalities, making use of the definition of $C(n; a)$ and subsequently reformulate the statements formulated with quantors into statements involving set inclusion. We introduce $Y = X \setminus \emptyset$ and exploit that $\emptyset \cap O = \emptyset$, so $\emptyset \subseteq [n] \setminus O\emptyset$ is a trivial condition. This proof setup will be used some more times for different but somewhat similar statements.

$$V(n; a; \emptyset; \emptyset; \emptyset) = \{ X \in C(n; a) : \forall i : i \notin X, \forall j : j \notin X \} = \bigcup_{k=1}^{m} \{ X \in \binom{[n]}{a_k} : X \subseteq [n] \setminus O, \emptyset \subseteq X \} = \bigcup_{k=1}^{m} \{ X \in \binom{[n]}{a_k} : \exists Y \subseteq [n] \setminus \emptyset : \emptyset \cup Y = X \subseteq [n] \setminus O \} = \bigcup_{k=1}^{m} \{ \emptyset \cup Y \in \binom{[n]}{a_k} : Y \subseteq [n] \setminus \emptyset ; \emptyset \cup Y \subseteq [n] \setminus O \} = \bigcup_{k=1}^{m} \{ \emptyset \cup Y \in \binom{[n]}{a_k} : Y \subseteq [n] \setminus (\emptyset \cup O) \} = \bigcup_{k=1}^{m} \{ \emptyset \cup Y : Y \in \binom{[n] \setminus (\emptyset \cup O)}{a_k - |\emptyset|} \} \quad \Box$$

This type of representation of a vertex set will appear more often from now on, as it allows us to recognize some patterns easily and allows us to prove many statements in similar ways. Let us first note a useful combinatorial equivalence, which will be used in many of those scenarios.

**Proposition 46.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $A \subseteq B \subseteq [n]$. Let

$$V = \bigcup_{k=k_{\min}}^{k_{\max}} \{ A \cup Y : Y \in \binom{B \setminus A}{a_k - |A|} \}$$

for some $1 \leq k_{\min} \leq k_{\max} \leq n$. Then:

$$\operatorname{conv}\{ x_X \in \mathbb{R}^n : X \in V \} \simeq P(|B| - |A|; (a_{k_{\min}} - |A|, \ldots, a_{k_{\max}} - |A|)).$$

**Proof.** We will prove the combinatorial equivalence by giving an affine isomorphy between the two. For ease of notation, let us simply call the right-hand side polytope $\varphi_P$. We recall the definition of a polytope of a cardinality homogeneous set system implying that the vertices of $P$ are exactly the incidence vectors of subsets of $|[B| - |A]|$ with a cardinality $a_k - |A|$ for some $k_{\min} \leq k \leq k_{\max}$. 

22
Let us fix an arbitrary injective map $\psi : \|B| - |A|\| \to [n]$ (there is one as $|B| - |A| \leq n$) and use it to define an $n \times (|B| - |A|)$-matrix $M$ by setting

$$M_{ij} := \begin{cases} 1, & i = \psi(j) \\ 0, & \text{else}. \end{cases}$$

Now the $r$-th row of $M$ is all-zero if and only if $r \not\in B \setminus A$, while the other rows contain the transposed $(|B| - |A|)$-dimensional unit vectors $e_1^p, \ldots, e_n^p$ in some order we shall not pay any attention to.

We observe that, for every vertex $v$ of $P$, the vector $Mv$ is an $n$-dimensional 0/1-vector with $(Mv)_i = 0$ whenever $i \not\in B \setminus A$. Moreover, $1^TMv = 1^Tv = a_k - |A|$ for some $k_{\min} \leq k \leq k_{\max}$.

Therefore, $Mv$ is the incidence vector of a $Y$-set and $Mv + \chi_A$ is the incidence vector of a set in $V$, as $A$ and $Y$ are disjoint. Conversely, if $Mv + \chi_A$ is the incidence vector of a set in $V$, then $Mv$ has to be the incidence vector of some $Y$-set. As $1^TMv = 1^Tv$, the entry sum of $v$ has to be equal to $a_k - |A|$ for some $k_{\min} \leq k \leq k_{\max}$, therefore $v$ is a vertex of $P$.

As the rank of $M$ is $|B| - |A|$, the associated affine transformation $\varphi(v) := Mv + \chi_A$ is a bijection between $\mathbb{R}^{|B| - |A|}$ (where $P$ lies) and the $(|B| - |A|)$-dimensional affine subspace of $\mathbb{R}^n$, $\{v \in \mathbb{R}^n : \forall i \in A : x_i = 1, \forall j \not\in B : x_j = 0\}$, where all the incidence vectors of $V$ and consequently, their convex hulls, are situated.

We conclude that $\varphi$ is a bijection between the vertices of $P$ and $\{\chi_X \in \mathbb{R}^n : X \in V\}$, where the images of the convex hulls are the convex hulls of the images; they are affinely isomorphic and combinatorially equivalent. \qed

We will now try to determine which configurations of in-set $\mathbb{I}$ and out-set $\mathbb{O}$ yield vertex sets of which class. Let us first make some observations on the cardinality of $\mathbb{I}$. We note that if $|\mathbb{I}| > a_m$, no $X \subseteq [n]$ with $|X| \leq a_m$ may satisfy $\mathbb{I} \subseteq X$ and consequently, $V(n; a; \emptyset; \mathbb{I}; \emptyset; \emptyset) = \emptyset$. We will therefore consider only those $\mathbb{I} \subseteq [n]$ with $|\mathbb{I}| \leq a_m$. We now distinguish the case where $\mathbb{I}$ is on a layer and the case where $\mathbb{I}$ is in a gap. In both cases, we define $i \in [m]$ to be the smallest number such that $|\mathbb{I}| \leq a_i$ and now categorise the possible cardinalities of $\mathbb{O}$ with respect to the aforementioned index $i$.

We observe that if $|[n] \setminus \mathbb{O}| < a_i$, no set of cardinality $\geq a_i$ may satisfy $X \subseteq [n] \setminus \mathbb{O}$, while by definition of $i$, no set of cardinality $< a_i$ may satisfy $\mathbb{I} \subseteq X$ and consequently, $V(n; a; \emptyset; \mathbb{I}; \emptyset; \emptyset) = \emptyset$. By a similar argument, if $|[n] \setminus \mathbb{O}| = a_i$, then $V(n; a; \emptyset; \mathbb{I}; \emptyset; \emptyset) = \{|[n] \setminus \mathbb{O}|\}$, a one-element-set which we are not interested in. We will therefore consider only those $\mathbb{O} \subseteq [n]$ with $\mathbb{I} \cap \mathbb{O}$ empty and $|[n] \setminus \mathbb{O}| \geq a_i$.

Let us list the layers and gaps the set $[n] \setminus \mathbb{O}$ may be in: If $m = i$, it has to be contained in gap $i$. If $m \geq i + 1$, it is either contained in gap $i$ or on layer $i + 1$ or in gap $i + 1$, while for $m \geq i + 2$, there is the additional possibility that $|[n] \setminus \mathbb{O}| \geq a_{i+1}$, so $[n] \setminus \mathbb{O}$ might be in gap or layer $j$ with $j \geq i + 2$. Table 4 and gives an overview of possible combinations. For each configuration, it contains the class of vertex sets generated. It will be our next step to reason these assignments.

### 4.4.1 Classification of the Vertex Sets

We will now specify class, vertex set, layer configuration, combinatorial type and dimension for each of the cases in Table 4 starting with the smallest vertex set.

**Proposition 47.** Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $\mathbb{I}, \mathbb{O} \subseteq [n]$ with $\mathbb{I} \cap \mathbb{O} = \emptyset$. \hfill \qed
| $|\|\| = a_i$ | vertex (Prop. 47.1) | 1-1-pyramid (Prop. 47.3) | 1-many-pyramid (Prop. 47.5) | many-layer (Prop. 47.7) |
| $|\| < a_i$ | full-1-layer (Prop. 47.2) | many-1-pyramid (Prop. 47.4) | full-2-layer (Prop. 47.6) | gap/layer $\geq i + 2$ |

| gap $i$ | layer $i + 1$ | gap $i + 1$ | |

Table 2: Classes for vertex sets for $M$ and $S$ both empty

1. If $|\| = a_i < |[n] \setminus O|$ for some $i \in [m]$ with ($i = m$ or $|[n] \setminus O| < a_{i+1}$), then $V(n; a; \emptyset; \emptyset; \emptyset)$ is a one-element set with

$$V(n; a; \emptyset; \emptyset; \emptyset) = \{\|\}.$$  

2. If $|\| < a_i < |[n] \setminus O|$ for some $i \in [m]$ with ($i = 1$ or $a_{i-1} < |\|$) and ($i = m$ or $|[n] \setminus O| < a_{i+1}$), then $V(n; a; \emptyset; \emptyset; \emptyset)$ is a full-1-layer set with

$$V(n; a; \emptyset; \emptyset; \emptyset) = \{\| \cup Y : Y \in \left( \left([n] \setminus (\| \cup O) \right) \setminus a_i - |\| \right) \}$$
$$L(n; a; \emptyset; \emptyset; \emptyset) = \{i\}$$
$$F(n; a; \emptyset; \emptyset; \emptyset) \cong \text{pyr}(P(n - |\| - |O|; (a_i - |\|))$$
$$d(n; a; \emptyset; \emptyset; \emptyset) = n - |\| - |O| - 1 \geq 1.$$  

3. If $|\| = a_i < a_{i+1} = |[n] \setminus O|$ for some $i \in [m - 1]$, then $V(n; a; \emptyset; \emptyset; \emptyset)$ is a 1-1-pyramid set with

$$V(n; a; \emptyset; \emptyset; \emptyset) = \{\|, [n] \setminus O\}$$
$$L(n; a; \emptyset; \emptyset; \emptyset) = \{i, i + 1\}$$
$$F(n; a; \emptyset; \emptyset; \emptyset) \cong \Delta_i$$
$$d(n; a; \emptyset; \emptyset; \emptyset) = 1.$$  

4. If $|\| < a_i < a_{i+1} = |[n] \setminus O|$ for some $i \in [m - 1]$ with ($i = 1$ or $a_{i-1} < |\|$), then $V(n; a; \emptyset; \emptyset; \emptyset)$ is a many-1-pyramid set with

$$V(n; a; \emptyset; \emptyset; \emptyset) = \{\| \cup Y : Y \in \left( \left([n] \setminus (\| \cup O) \right) \setminus a_i - |\| \right) \} \cup \{[n] \setminus O\}$$
$$L(n; a; \emptyset; \emptyset; \emptyset) = \{i, i + 1\}$$
$$F(n; a; \emptyset; \emptyset; \emptyset) \cong \text{pyr}(P(n - |\| - |O|; (a_i - |\|))$$
$$d(n; a; \emptyset; \emptyset; \emptyset) = n - |\| - |O| \geq 2.$$  

5. If $|\| = a_i < a_{i+1} < |[n] \setminus O|$ for some $i \in [m - 1]$ with ($i + 1 = m$ or $|[n] \setminus O| < a_{i+2}$), then $V(n; a; \emptyset; \emptyset; \emptyset)$ is a 1-many-pyramid set with

$$V(n; a; \emptyset; \emptyset; \emptyset) = \{\| \cup \{\| \cup Y : Y \in \left( \left([n] \setminus (\| \cup O) \right) \setminus a_{i+1} - |\| \right) \} \}$$
propositions and constructions. We will therefore just give proofs for the special case \(1\), where

\[
\text{Proof of some cases of Prop. 47.}
\]

\[
\text{The proofs for the portions of this proposition are quite similar and use the same reformulations,}
\]

\[
\text{propositions and constructions. We will therefore just give proofs for the special case 1, where}
\]

\[
\text{reasoning among the remaining cases.}
\]

\[
\text{6. If } |\mathbb{I}| < a_i < a_{i+1} < |\{n\} \setminus \mathbb{O}| \text{ for some } i \in [m-1] \text{ with } (i = 1 \text{ or } a_{i-1} < |\mathbb{I}|) \text{ and } (i+1 = m \text{ or } |\{n\} \setminus \mathbb{O}| < a_{i+2}), \text{ then } V(n; a; \emptyset; \emptyset; \emptyset) \text{ is a full-2-layer set with}
\]

\[
V(n; a; \emptyset; \emptyset; \emptyset) = \{ \emptyset \cup Y : Y \in \left( [n] \setminus (\emptyset \cup \mathbb{O}) \right) \cup \left( [n] \setminus (\emptyset \cup \mathbb{O}) \right) \}
\]

\[
L(n; a; \emptyset; \emptyset; \emptyset) = \{i, i+1\}
\]

\[
F(n; a; \emptyset; \emptyset; \emptyset) \simeq P(n - |\mathbb{I}| - |\mathbb{O}|; (a_i - |\mathbb{I}|, a_{i+1} - |\mathbb{I}|))
\]

\[
d(n; a; \emptyset; \emptyset; \emptyset) = n - |\mathbb{I}| - |\mathbb{O}| \geq 3.
\]

\[
\text{7. If } |\mathbb{I}| \leq a_i < a_{i+1} < a_{i+2} \leq |\{n\} \setminus \mathbb{O}| \text{ for some } i \in [m-2], \text{ define } L := \{ i \in [m] : |\mathbb{I}| \leq a_i \leq |\{n\} \setminus \mathbb{O}| \}, t := \min(L), u := \max(L) \text{ and define the cardinality sequence } b
\]

\[
b_j := a_{j+1} - |\mathbb{I}| \text{ for } j \in [u - t + 1]. \text{ Then } V(n; a; \emptyset; \emptyset; \emptyset) \text{ is a many-layer set with}
\]

\[
V(n; a; \emptyset; \emptyset; \emptyset) = \bigcup_{k=1}^{m} \{ \emptyset \cup Y : Y \in \left( [n] \setminus (\emptyset \cup \mathbb{O}) \right) \}
\]

\[
L(n; a; \emptyset; \emptyset; \emptyset) = B \supseteq \{i, i+1, i+2\}
\]

\[
F(n; a; \emptyset; \emptyset; \emptyset) \simeq P(n - |\mathbb{I}| - |\mathbb{O}|; b)
\]

\[
d(n; a; \emptyset; \emptyset; \emptyset) = n - |\mathbb{I}| - |\mathbb{O}| \geq 2.
\]

The proofs for the portions of this proposition are quite similar and use the same reformulations, propositions and constructions. We will therefore just give proofs for the special case 1, where \(V(n; a; \emptyset; \emptyset; \emptyset)\) is a one-element set; and for case 4 which arguably requires the most elaborate reasoning among the remaining cases.

**Proof of some cases of Prop. 47.**

1. The one-element-case:

   In this case, \(|\mathbb{I}| = a_i < |\{n\} \setminus \mathbb{O}|\) for some \(i \in [m]\) with \((i = m \text{ or } |\{n\} \setminus \mathbb{O}| < a_{i+1})\). As

   \(|\mathbb{I}| = a_i\), no set \(X\) of cardinality \(|X| < a_i\) may satisfy \(\mathbb{I} \subseteq X\). If \(i = m\), there are no sets in \(C(n; a)\) of cardinality \(a_i < |X|\). If \(|\{n\} \setminus \mathbb{O}| < a_{i+1}\), no set of cardinality \(a_{i+1} \leq |X|\) satisfies \(\mathbb{I} \subseteq [n] \setminus \mathbb{O}\). We may conclude now that \(V(n; a; \emptyset; \emptyset; \emptyset) = \{X \in C(n; a) : \mathbb{I} \subseteq [n] \setminus \mathbb{O}\} \) can only contain sets of cardinality \(a_i\). As \(\mathbb{I}'s\) only superset of cardinality \(a_i\) is \(\mathbb{I}\) itself, which is a subset of \([n] \setminus \mathbb{O}\) as \(\mathbb{I} \cap \mathbb{O}\) is empty, we deduce the desired equality.

4. The many-1-pyramid case:

   In this case, \(|\mathbb{I}| < a_i < a_{i+1} = |\{n\} \setminus \mathbb{O}|\) for some \(i \in [m-1]\) with \((i = 1 \text{ or } a_{i-1} < |\mathbb{I}|)\). We recall Proposition 45 stating that

   \[
   V(n; a; \emptyset; \emptyset; \emptyset) = \bigcup_{k=1}^{m} \{ \emptyset \cup Y : Y \in \left( [n] \setminus (\emptyset \cup \mathbb{O}) \right) \}
   \]

   As we know \(i = 1 \text{ or } a_{i-1} < |\mathbb{I}|\) to hold, we deduce that non-negative values for \(a_k - |\mathbb{I}|\) occur only if \(k \geq i\). On the other hand, as \(a_{i+1} = |\{n\} \setminus \mathbb{O}|\), we see that \(a_k - |\mathbb{I}| \leq

   25
We see an example in Figure 5 on the next page.

As for the latter case, we notice that because of $|\mathcal{I}| < |\mathcal{O}|$ implies that there is more than one feasible choice for $Y$ in the first portion, while there clearly is exactly one set on layer $a_{i+1}$. Thus, $V(n; a; \emptyset; \emptyset)$ satisfies the definition of a many-1-pyramid set and

$$L(n; a; \emptyset; \emptyset) = \{i, i+1\}.$$
Figure 5: Some vertex sets for $n = 5, a = (1, 4)$, empty $M$ and $S$
4.4.2 Uniqueness of Representation

In order to use our findings to count the faces of \( P(n; a) \), it will be useful that different in- or out-sets \( \mathbb{I}, \mathbb{O} \) always yield different vertex sets (except for the empty set and single-vertex sets). For that, we will formulate a more general statement which will be useful for other cases as well.

**Proposition 48.** Let \( n \in \mathbb{N}, k \in \mathbb{Z}, 0 \leq k \leq n \) and \( A_1, A_2, B_1, B_2 \subseteq [n] \) with \( A_1 \subseteq B_1, A_2 \subseteq B_2 \) and \( |A_1|, |A_2| < k < |B_1|, |B_2| \). For \( i \in \{1, 2\} \), define

\[
V_i := \{ A_1 \cup Y : Y \in \left( B_i \setminus A_1 \right) \}.
\]

Then the following statement holds:

\[
V_1 = V_2 \iff A_1 = A_2 \land B_1 = B_2
\]

**Proof.** The right-to-left direction is a trivial implication. For the left-to-right direction, we will use contraposition. Let us first notice that \( V_1, V_2 \) are non-empty: \( |A_1| < k < |B_1| \Rightarrow 0 < k - |A_1| < |B_1| - |A_1| \Rightarrow \left( B_i \setminus A_1 \right) \neq \emptyset \), as \( A_1 \subseteq B_i \) implies \( |B_i \setminus A_1| = |B_i| - |A_i| \), and therefore \( V_i \neq \emptyset \).

We will now show that if \( A_1 \neq A_2 \) or \( B_1 \neq B_2 \), the left-hand-side statement is invalid.

- If \( A_1 \neq A_2 \), without loss of generality we may assume that \( A_1 \setminus A_2 \neq \emptyset \), so let \( j \in A_1 \setminus A_2 \). On one hand, \( j \in A_1 \) implies that \( \forall X \in \{ A_1 \cup Y : Y \in \left( B_2 \setminus A_1 \right) \} : j \notin X \) and therefore, \( V_1 \neq V_2 \) as these \( \forall \)-statements over non-empty sets are contradictory.

- If \( j \in B_2 \setminus A_2 \), we take a look at \( \left( B_2 \setminus \left( A_2 \cup \{ j \} \right) \right) \) and see that it is non-empty as \( 0 < k - |A_2| < |B_2| - |A_2| \Rightarrow 0 < k - |A_2| \leq |B_2 \setminus \left( A_2 \cup \{ j \} \right)| \). So we may pick \( Z \in \left( B_2 \setminus \left( A_2 \cup \{ j \} \right) \right) \), then \( Z \in V_2, j \notin Z \) which would contradict our finding about \( V_1 \) from above, therefore \( V_1 \neq V_2 \).

- If \( B_1 \neq B_2 \), without loss of generality we may assume that \( B_1 \setminus B_2 \neq \emptyset \), so let \( j \in B_1 \setminus B_2 \). On one hand, \( j \notin B_2 \) implies that \( \forall X \in \{ A_2 \cup Y : Y \in \left( B_2 \setminus A_1 \right) \} : j \notin X \) and therefore, \( V_1 \neq V_2 \) as these \( \forall \)-statements over non-empty sets are contradictory.

- If \( j \in B_1 \setminus A_1 \), we take a look at \( \left( B_1 \setminus \left( A_1 \cup \{ j \} \right) \right) \) and see that it is non-empty as \( 0 < k - |A_1| < |B_1| - |A_1| \Rightarrow 0 \leq k - |A_1| - 1 < |B_1 \setminus \left( A_1 \cup \{ j \} \right)| \). So we may pick \( Z \in \left( B_1 \setminus \left( A_1 \cup \{ j \} \right) \right) \), then \( Z \cup \{ j \} \in \left( B_1 \setminus \left( A_1 \cup \{ j \} \right) \right) = A_1 \cup Z \cup \{ j \} \in V_1 \) which would contradict our finding about \( V_2 \) from above, therefore \( V_1 \neq V_2 \).

\[ \square \]

Let us now appropriate this general result to the cases of vertex sets \( V(n; a; M; I; O; S) \) with empty \( M \) and \( S \) to show that two such vertex sets are distinct unless they agree in both the in-set \( I \) and the out-set \( O \), except maybe for vertex sets containings less than two elements.
Proposition 49. Let $n \in \mathbb{N}$ and $a$ be a cardinality sequence. Let $I_1, I_2, O_1, O_2 \subseteq [n], I_1 \cap O_1 = I_2 \cap O_2 = \emptyset$. If $|V(n; a; \emptyset; I_1; O_1; \emptyset)| = |V(n; a; \emptyset; I_2; O_2; \emptyset)| > 1$, then:

$$V(n; a; \emptyset; I_1; O_1; \emptyset) = V(n; a; \emptyset; I_2; O_2; \emptyset) \iff I_1 = I_2 \land O_1 = O_2$$

Proof. The right-to-left direction is a trivial implication. For the left-to-right direction, which we will prove by contraposition, let us first consider the case where $V(n; a; \emptyset; I_1; O_1; \emptyset)$ is a 1-1-pyramid. We learned in Proposition 47 that in this case, $V(n; a; \emptyset; I_1; O_1; \emptyset) = \{I_1, [n] \setminus O_1\}$. Consequently, $V(n; a; \emptyset; I_1; O_1; \emptyset) = V(n; a; \emptyset; I_2; O_2; \emptyset)$ holds if and only if $\{I_1, [n] \setminus O_1\} = \{I_2, [n] \setminus O_2\}$. This cannot be achieved by $I_1 = [n] \setminus O_2 \land [n] \setminus O_1 = I_2$, because in that case, $I_2 \cap O_2 = ([n] \setminus O_1) \cap ([n] \setminus I_1) = [n] \setminus (I_1 \cup O_1)$. This set is non-empty because otherwise $V(n; a; \emptyset; I_1; O_1; \emptyset)$ would not have fallen into the class of a 1-1-pyramid, so $I_2 \cap O_2 \neq \emptyset$ in contradiction to our assumption. Therefore we may deduce $I_1 = I_2 \land O_1 = O_2$ and our desired statement is true for 1-1-pyramids.

We may now think about vertex sets of at least two vertices which are not 1-1-pyramids. In that case, there is a layer $k \in [n]$ that contains at least two of the vertices. We learned from Proposition 45 that the set of vertices of $V(n; a; \emptyset; I_1; O_1; \emptyset)$ in that layer is equal to $\{I_1 \cup Y : Y \in \{a_k - |I_1|\}\}$, where the set having at least two elements implies that $0 < a_k - |I_1| < n - |I_1| - |O_1|$. Of course, these statements still hold with 2’s as indices instead of 1’s, so with $A_1 := I_1, A_2 := I_2, B_1 := [n] \setminus O_1, B_2 := [n] \setminus O_2$, we may apply Proposition 48 so see that $I_1 \neq I_2 \lor O_1 \neq O_2 \Rightarrow A_1 \neq A_2 \lor B_1 \neq B_2$, so $V(n; a; \emptyset; I_1; O_1; \emptyset)$ and $V(n; a; \emptyset; I_2; O_2; \emptyset)$ don’t contain the same vertices on layer $k$ and therefore are not identical.

Let us now handle the next configuration: the vertex sets generated by empty $S$ and either $M = \{\text{min}\}$ or $M = \{\text{max}\}$.

### 4.5 Vertex Sets for $M = \{\text{min}\}$ or $M = \{\text{max}\}$; and $S = \emptyset$

For this configuration, we may directly determine the combinatorial type and the dimension of a face of this form: It is (in the general case) a lower-dimensional polytype of a cardinality homogeneous set system, the dimension of which is determined by the number of elements in $I$ and $O$.

Proposition 50. Let $n \in \mathbb{N}$, $a$ be a cardinality sequence. Let $I, O \subseteq [n]$ with $I \cap O = \emptyset$ and $M = \{\text{min}\}$ or $M = \{\text{max}\}$. We set $k = 1$ if $M = \{\text{min}\}$ and $k = m$ otherwise.

1. If $|I| \geq a_k$ or $a_k \leq |n| - |O|$, then

$$|V(n; a; M; I, O; \emptyset)| \leq 1.$$

2. If $|I| < a_k < n - |O|$, then $V(n; a; M; I, O; \emptyset)$ is a full-1-layer set with

$$V(n; a; M; I, O; \emptyset) = \{\emptyset \cup Y : Y \in \left\{\left[\left[2 \left(\left|\left[\{n\} \setminus (I \cup O)\}\right|\right)\right)\right]\right\}

L(n; a; M; I, O; \emptyset) = \{k\}

F(n; a; M; I, O; \emptyset) \simeq P(n - |I| - |O|; (a_k - |I|))

d(n; a; M; I, O; \emptyset) = n - |I| - |O| - 1 \geq 1.$$
**Proof.** As for the reformulation of the vertex set, we will follow an adapted and shortened version of the proof of Prop. 45. Within the definition of $V(n; a; M; I; O; \emptyset)$, we find that $\forall S \in \emptyset : X \in V_{CF}(S)\emptyset$ is a trivial condition. We then introduce once more $Y := X \setminus I$ and exploit that $I \cap O = \emptyset$, so $\|I\| \subseteq [n] \setminus O\emptyset$ is another trivial condition.

$$V(n; a; M; I; O; \emptyset) = \{X \in \binom{[n]}{a_k} : \forall S \in \emptyset : X \in V_{CF}(S), I \subseteq X \subseteq [n] \setminus O\}$$

$$= \{X \in \binom{[n]}{a_k} : I \subseteq X, X \subseteq [n] \setminus O\}$$

$$= \{I \cup Y : Y \subseteq [n] \setminus (I \cup O)\}$$

$$= \{I \cup Y : Y \in \binom{[n] \setminus (I \cup O)}{a_k - \|I\|}\}$$

This reformulation implies that the cardinality of the set is equal to $\binom{n - \|I\| - |O|}{a_k - \|I\|}$, which is larger than one if and only if $\|I\| < a_k < n - |O|$, which proves the Proposition’s first part. From here on, let us assume that the in- and out-set satisfy this strict inequality. If they do, the vertex set contains at least two elements on layer $k$ and none on any other layer – consequently, $L(n; a; M; I; O; \emptyset) = \{k\}$. As for the combinatorial type of the generated face, we recognise the stated result to be an immediate consequence of Proposition 46. As the precondition $\|I\| < a_k < n - |O|$ implies that $0 < a_k < n$, Proposition 36 tells us that the dimension of this combinatorially equivalent polytope is $n - \|I\| - |O| - 1$. Moreover, the same precondition implies that $n - |O| - \|I\| \geq 2$, thus, the dimension of the face is at least 1.

We observe that for $m = 1$, $M = \{\min\}$ and $M = \{\max\}$ are equivalent. Our formulations take that into account. However, in the case that $m = 1$, all the faces of $P(n; a)$ can be represented by configurations with empty $M$ anyway. In our example of $C(5; (1, 4))$, let us have a look at one vertex set in this configuration for $M = \{\min\}$ and $M = \{\max\}$ each, depicted in Figure 6. We observe that neither of these vertex sets could have come out of a configuration with empty $M$.

![Figure 6: Some vertex sets for $n = 5$, $a = (1, 4)$ empty $S$ and either $M = \{\min\}$ or $M = \{\max\}$](image)

**Observation 51.** We notice that the uniqueness criterion provided by Prop. 48 is directly applicable to this configuration of vertex sets, so for $I, O \subseteq [n]$ with $I \cap O = \emptyset$ and either $M = \{\min\}$ or...
\( M = \{ \max \}, \)

\[
V(n; a; M; \emptyset ; \emptyset) = V(n; a; M; I_2; \emptyset) \iff I_1 = I_2 \wedge O_1 = O_2.
\]

### 4.6 Vertex Sets for \( M = \emptyset \) and \( S \) a CF-family in a gap

For this last configuration of vertex sets, we have to be most careful, because having many hyperplanes involved, some special cases occur that need to be treated with caution. We start with a basic statement about the vertex sets of this configuration and use that first result to explore some cases and deduce more detailed statements.

**Proposition 52.** Let \( n \in \mathbb{N}, a \) a cardinality sequence and \( I, O \subseteq [n], I \cap O = \emptyset \). Let \( S \subseteq S(n; a) \) be a CF-family in gap \( g \in [m - 1] \). Then:

\[
V(n; a; \emptyset; I; O; S) = \{ X \in \binom{[n]}{a_g} : I \subseteq X \subseteq S^c \} \cup \{ X \in \binom{[n]}{a_{g+1}} : I \cup S^j \subseteq X \subseteq [n] \}.
\]

**Proof.** We plug in the definition, apply Proposition 31 for a CF-family in gap \( g \) and consider the two halves of the vertex set separately.

\[
V(n; a; \emptyset; I; O; S) = \bigcup_{k=1}^{m} \binom{[n]}{a_k} : \forall S \in S: X \in V_{CF}(S), I \subseteq X \subseteq [n] \}
\]

\[
\cap \{ X \in \bigcup_{k=1}^{m} \binom{[n]}{a_k} : I \subseteq X \subseteq [n] \}
\]

\[
\cap \{ X \in \bigcup_{k=1}^{m} \binom{[n]}{a_k} : I \subseteq X \subseteq [n] \}
\]

\[
\cap \{ X \in \bigcup_{k=1}^{m} \binom{[n]}{a_k} : I \subseteq X \subseteq [n] \}
\]

\[
\cup \{ X \in \bigcup_{k=1}^{m} \binom{[n]}{a_k} : I \subseteq X \subseteq [n] \}
\]

\[
\cup \{ X \in \bigcup_{k=1}^{m} \binom{[n]}{a_k} : I \subseteq X \subseteq [n] \}
\]

\[
\begin{align*}
\text{With the help of this first result, we may take a closer look at each of the two disjoint parts before discussing their union. We first name some constraints under which one of the parts is empty and determine conditions under which there is more than one vertex contained in a part.}

\textbf{Proposition 53.} Let \( n \in \mathbb{N}, a \) be a cardinality sequence and \( I, O \subseteq [n] \) with \( I \cap O = \emptyset \) and \( S \subseteq S(n; a) \) a CF-family in gap \( g \in [m - 1] \). Then:
\end{align*}
\]
1. If \( \mathbb{I} \not\subseteq \mathbb{S} \) or \( |\mathbb{I}| > a_g \) or \( a_g > |\mathbb{S} \setminus \mathbb{O}| \), then
\[
\{ X \in \binom{[n]}{a_g} : \mathbb{I} \subseteq X \subseteq \mathbb{S} \setminus \mathbb{O} \} \text{ is empty.}
\]

2. If \( \mathbb{S} \cup \not\subseteq [n] \setminus \mathbb{O} \) or \( |\mathbb{I} \cup \mathbb{S} \cup| > a_{g+1} \) or \( a_{g+1} > |[n] \setminus \mathbb{O}| \), then
\[
\{ X \in \binom{[n]}{a_{g+1}} : \mathbb{I} \cup \mathbb{S} \cup \subseteq X \subseteq [n] \setminus \mathbb{O} \} \text{ is empty.}
\]

3. If \( \mathbb{I} \subseteq \mathbb{S} \) and \( |\mathbb{I}| \leq a_g \leq |\mathbb{S} \setminus \mathbb{O}| \), then
\[
\{ X \in \binom{[n]}{a_g} : \mathbb{I} \subseteq X \subseteq \mathbb{S} \setminus \mathbb{O} \} = \{ \mathbb{I} \cup \mathbb{Y} : \mathbb{Y} \in \binom{\mathbb{S} \setminus (\mathbb{I} \cup \mathbb{O})}{a_g - |\mathbb{I}|} \}
\]
is non-empty. It contains at least two elements if and only if \( |\mathbb{I}| < a_g < |\mathbb{S} \setminus \mathbb{O}| \).

4. If \( \mathbb{S} \cup \not\subseteq [n] \setminus \mathbb{O} \) and \( |\mathbb{I} \cup \mathbb{S} \cup| \leq a_{g+1} \leq |[n] \setminus \mathbb{O}| \), then
\[
\{ X \in \binom{[n]}{a_{g+1}} : \mathbb{I} \cup \mathbb{S} \cup \subseteq X \subseteq [n] \setminus \mathbb{O} \} = \{ (\mathbb{I} \cup \mathbb{S} \cup) \cup \mathbb{Y} : \mathbb{Y} \in \binom{[n] \setminus (\mathbb{I} \cup \mathbb{O} \cup \mathbb{S} \cup)}{a_{g+1} - |\mathbb{I} \cup \mathbb{S} \cup|} \}
\]
is non-empty. It contains at least two elements if and only if \( |\mathbb{I} \cup \mathbb{S} \cup| < a_{g+1} < |[n] \setminus \mathbb{O}| \).

**Proof.** For the first two cases, we just need to reason that no \( X \) may satisfy the requirements to be contained in the set, while for the latter two, we need to prove the stated equality and examine the number of elements contained depending on the strictness of the cardinality inequalities.

1. If \( \mathbb{I} \not\subseteq \mathbb{S} \), then \( \mathbb{I} \subseteq X \) implies \( X \not\subseteq \mathbb{S} \setminus \mathbb{O} \). If \( |\mathbb{I}| > a_g \), then \( \mathbb{I} \subseteq X \) implies \( |X| > a_g \). If \( a_g > |\mathbb{S} \setminus \mathbb{O}| \), then \( X \subseteq \mathbb{S} \setminus \mathbb{O} \) implies \( |X| < a_g \). In each of these cases, \( \{ X \in \binom{[n]}{a_g} : \mathbb{I} \subseteq X \subseteq \mathbb{S} \setminus \mathbb{O} \} \) is empty.

2. If \( \mathbb{S} \cup \not\subseteq [n] \setminus \mathbb{O} \), then \( \mathbb{I} \cup \mathbb{S} \cup \subseteq X \) implies \( X \not\subseteq [n] \setminus \mathbb{O} \). If \( |\mathbb{I} \cup \mathbb{S} \cup| > a_{g+1} \), then \( \mathbb{I} \cup \mathbb{S} \cup \subseteq X \) implies \( |X| > a_{g+1} \). If \( a_{g+1} > |[n] \setminus \mathbb{O}| \), then \( X \subseteq [n] \setminus \mathbb{O} \) implies \( |X| < a_{g+1} \). In each of these cases, \( \{ X \in \binom{[n]}{a_{g+1}} : \mathbb{I} \cup \mathbb{S} \cup \subseteq X \subseteq [n] \setminus \mathbb{O} \} \) is empty.

3. In the case where \( \mathbb{I} \subseteq \mathbb{S} \) and \( |\mathbb{I}| \leq a_g \leq |\mathbb{S} \setminus \mathbb{O}| \), we conclude \( \mathbb{I} \subseteq \mathbb{S} \setminus \mathbb{O} \) to be true as \( \mathbb{I} \cap \mathbb{O} = \emptyset \). This allows us to introduce the reformulation of our established style:
\[
\{ X \in \binom{[n]}{a_g} : \mathbb{I} \subseteq X \subseteq \mathbb{S} \setminus \mathbb{O} \} = \{ \mathbb{I} \cup \mathbb{Y} : \mathbb{Y} \in \binom{\mathbb{S} \setminus (\mathbb{I} \cup \mathbb{O})}{a_g - |\mathbb{I}|} \}
\]

To prove the latter set’s non-emptiness, we observe that \( \mathbb{I} \subseteq \mathbb{S} \setminus \mathbb{O} \) implies \( |\mathbb{S} \setminus (\mathbb{I} \cup \mathbb{O})| = |\mathbb{S} \setminus \mathbb{O}| - |\mathbb{I}| \). Together with the precondition \( |\mathbb{I}| \leq a_g \leq |\mathbb{S} \setminus \mathbb{O}| \), we see that \( 0 \leq a_g - |\mathbb{I}| \leq |\mathbb{S} \setminus (\mathbb{I} \cup \mathbb{O})| \). This implies that \( \binom{\mathbb{S} \setminus (\mathbb{I} \cup \mathbb{O})}{a_g - |\mathbb{I}|} \) is non-empty. It contains more than one element if and only if both inequalities are strict, which is equivalent to \( |\mathbb{I}| < a_g < |\mathbb{S} \setminus \mathbb{O}| \). The cardinality of the entire set is identical, as \( \mathbb{I} \) and \( \mathbb{Y} \) are always disjoint.
4. In the case where \( S^U \subseteq [n] \setminus \emptyset \) and \(|\emptyset \cup S^U| \leq a_{g+1} \leq |[n] \setminus \emptyset|\), we may conclude \( \emptyset \cup S^U \subseteq [n] \setminus \emptyset \) to be true as \( \emptyset \cap \emptyset = \emptyset \) and therefore we may reformulate once more:

\[
\{ X \in \binom{[n]}{a_{g+1}} : \emptyset \cup S^U \subseteq X \subseteq [n] \setminus \emptyset \} = \{ (\emptyset \cup S^U) \cup Y : Y \in \binom{[n] \setminus (\emptyset \cup S^U)}{a_{g+1} - |(\emptyset \cup S^U)|} \}.
\]

To prove non-emptiness, we observe that here, the preconditions yield \( 0 \leq a_{g+1} - |(\emptyset \cup S^U)| \leq |[n] \setminus (\emptyset \cup S^U)|\), therefore \( \binom{[n] \setminus (\emptyset \cup S^U)}{a_{g+1} - |(\emptyset \cup S^U)|} \) is non-empty and so is the set constructed from it by disjointly uniting each element with \((\emptyset \cup S^U)\). Once more, the set contains at least two elements if and only if one has a real choice in the binomial coefficient, which is the case whenever \( 0 < a_{g+1} - |(\emptyset \cup S^U)| < |[n] \setminus \emptyset| - |(\emptyset \cup S^U)| \) holds with strict inequalities.

4.6.1 Classification of the Vertex Sets

We have seen how we may determine the number of vertices in each of the two parts. Let us now put the two parts back together and note some facts about the different cases. Distinguishing the three configurations »empty«, »one vertex« and »many vertices« for each of the two parts yields the nine cases arranged in Table 3 which we will discuss next.

<table>
<thead>
<tr>
<th>Number of vertices on layer ( i )</th>
<th>Number of vertices on layer ( i + 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>\emptyset</td>
</tr>
<tr>
<td>(Prop. 54.1)</td>
<td>(Prop. 54.3)</td>
</tr>
<tr>
<td>vertex</td>
<td>vertex</td>
</tr>
<tr>
<td>(Prop. 54.2)</td>
<td>(Prop. 54.5)</td>
</tr>
<tr>
<td>1</td>
<td>1-1-pyramid</td>
</tr>
<tr>
<td>(Prop. 54.4)</td>
<td>1-many-pyramid</td>
</tr>
<tr>
<td>\geq 2</td>
<td>full-1-layer</td>
</tr>
<tr>
<td>(Prop. 54.8)</td>
<td>(Prop. 54.1)</td>
</tr>
<tr>
<td>full-1-layer</td>
<td>many-1-pyramid</td>
</tr>
<tr>
<td>(Prop. 54.6)</td>
<td>(Prop. 54.7)</td>
</tr>
<tr>
<td>\geq 2</td>
<td>full-2-layer</td>
</tr>
<tr>
<td>(Prop. 54.9)</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Classes of vertex sets for \( M \) empty, \( S \) a CF-family in gap \( i \)

As we saw in the last proposition, we can read the number of elements in each part off the cardinality sequence and the sets \( I, O, S \). We already found out some facts about each of them, the classification will be useful for what the next proposition and its proof will be about: Discussing each class and examining the vertex set, the layer structure and the associated face’s combinatorial type and dimension. We will plug in the if-and-only-if statements for the number of elements in each part we found in the previous proposition and consider all combinations thereof.

**Proposition 54.** Let \( n \in \mathbb{N} \), \( a \) be a cardinality sequence and \( I, O \subseteq [n], I \cap O = \emptyset \) and \( S \subseteq S(n; a) \) a CF-family in gap \( g, g \in [m - 1] \).

1. If \( (I \not\subseteq S^O \lor |I| > a_g \lor a_g > |S^O \setminus O|) \) and \((S^U \not\subseteq [n] \setminus O \lor |I \cup S^U| > a_{g+1} \lor a_{g+1} > |[n] \setminus O|)\), then \( V(n; a; \emptyset; I; O; S) \) is empty.

2. If \( I \subseteq S^O \) and \( |I| \leq a_g \leq |S^O \setminus O| \) with \( a_g \in \{|I|, |S^O \setminus O|\} \) and \((S^U \not\subseteq [n] \setminus O \lor |I \cup S^U| > a_{g+1} \lor a_{g+1} > |[n] \setminus O|)\), then \( V(n; a; \emptyset; I; O; S) \) contains exactly one element.

3. If \( (I \not\subseteq S^O \lor |I| > a_g \lor a_g > |S^O \setminus O|) \) and \((S^U \not\subseteq [n] \setminus O \lor |I \cup S^U| \leq a_{g+1} \leq |[n] \setminus O|) \) with \( a_{g+1} \in \{|I \cup S^U|, |[n] \setminus O|\} \), then \( V(n; a; \emptyset; I; O; S) \) contains exactly one vertex.
4. If $I \subseteq \mathbb{S}^\cap$ and $|I| \leq a_g \leq |\mathbb{S}^\cap \setminus \mathbb{O}|$ with $a_g \in \{I, |\mathbb{S}^\cap \setminus \mathbb{O}|\}$ and $\mathbb{S}^U \subseteq [n] \setminus \mathbb{O}$, $|I \cup \mathbb{S}^U| \leq a_{g+1} \leq |[n] \setminus \mathbb{O}|$ with $a_{g+1} \in \{I \cup [n] \setminus \mathbb{O}, |[n] \setminus \mathbb{O}|\}$, we define

$$X_g := \begin{cases} I, & |I| = a_g \text{ and } X_{g+1} := \begin{cases} \mathbb{O}, & a_{g+1} = |[n] \setminus \mathbb{O}| \\ [n] \setminus \mathbb{S}^U, & \text{else.} \end{cases} \end{cases}$$

Then:

$$V(n; a; \emptyset; I; \mathbb{O}; \mathbb{S}) = V(n; a; \emptyset; X_g; X_{g+1}; \emptyset).$$

5. If $I \not\subseteq \mathbb{S}^\cap$ or $|I| > a_g$ or $a_g > |\mathbb{S}^\cap \setminus \mathbb{O}|$ and $\mathbb{S}^U \subseteq [n] \setminus \mathbb{O}$, $|I \cup \mathbb{S}^U| < a_{g+1} < |[n] \setminus \mathbb{O}|$, then $V(n; a; \emptyset; I \cup \mathbb{S}^U; \mathbb{O}; \{I \cup \mathbb{S}^U\})$ is a full-1-layer set with

$$V(n; a; \emptyset; I; \mathbb{O}; \mathbb{S}) = V(n; a; \emptyset; I \cup \mathbb{S}^U; \mathbb{O}; \{I \cup \mathbb{S}^U\})$$

$$= \{(I \cup \mathbb{S}^U) \cup Y : Y \in ([n] \setminus (I \cup \mathbb{O} \cup \mathbb{S}^U)) \}$$

$$L(n; a; \emptyset; I; \mathbb{O}; \mathbb{S}) = \{g + 1\}$$

$$F(n; a; \emptyset; I; \mathbb{O}; \mathbb{S}) \approx P(n - |\mathbb{O}| - |I \cup \mathbb{S}^U|; (a_{g+1} - |I \cup \mathbb{S}^U|))$$

$$d(n; a; \emptyset; I; \mathbb{O}; \mathbb{S}) = n - |\mathbb{O}| - |I \cup \mathbb{S}^U| - 1.$$
Proof of some cases of Prop. 54.

Before we turn to the proof, let us take a look at some examples. Vertex sets for all the interesting cases 4-9 can be found in Table 7 on the following page. The sets contained in the CF-family, of a cardinality homogeneous set system nor a pyramid over one, but a join of two such polytopes. Furthermore, we will prove case 9, which is to a large extent archetypical for cases 5-8 with the peculiarity that the combinatorial type of the associated face is neither a lower-dimensional polytope of a cardinality homogeneous set system nor a pyramid over one, but a join of two such polytopes.

We will first give a proof for case 4, providing an equal vertex set with empty CF-family. In the first three cases, the statement follows directly from splitting up the vertex set as done in Proposition 52 and then reading off the number of vertices in each part given by Proposition 53.

We will first give a proof for case 4, providing an equal vertex set with empty CF-family. In addition, we will prove case 9, which is to a large extent archetypical for cases 5-8 with the peculiarity that the combinatorial type of the associated face is neither a lower-dimensional polytope of a cardinality homogeneous set system nor a pyramid over one, but a join of two such polytopes. Before we turn to the proof, let us take a look at some examples. Vertex sets for all the interesting cases 4-9 can be found in Table 7 on the following page. The sets contained in the CF-family, which is denoted in the condensed notation containing as few sets as necessary, are coloured in red.

**Proof of some cases of Prop. 54**

4. We learned in Proposition 52 that

\[
V(n; a; \emptyset; \emptyset; \emptyset; \emptyset; \emptyset) = \{ X : X \subseteq [n] \setminus \emptyset \}.
\]

The combination of the preconditions \( \emptyset \subseteq S^\gamma \setminus \emptyset \subseteq a_g \subseteq [n] \setminus \emptyset \) and \( a_g \in \{ \emptyset \}, \emptyset \subseteq [n] \setminus \emptyset \) allows us to apply Prop. 53 and see that the former set is equal to \( \{ X_g \} \) by definition. Likewise, the remaining preconditions imply that the latter set is equal to \( \{ [n] \setminus X_{g+1} \} \).

Furthermore, \( X_g \subseteq [n] \setminus \emptyset \subseteq [n] \setminus \emptyset \subseteq X_{g+1} \), so \( X_g \) and \( X_{g+1} \) are disjoint, making \( V(n; a; \emptyset; X_g; X_{g+1}; \emptyset) \) a feasible configuration for empty \( S \). As \( |X_g| = a_g \) and

\[
|X_{g+1}| = a_g + 1.
\]
Figure 7: Some vertex sets for $n = 5$, $a = (1, 4)$, empty $M$ and $S$ a CF-family in gap 1
of Prop. 48, which gave us a criterion as to when two sets of vertices on some layer are identical.

9. The preconditions for this case are therefore we know the dimension of a free join of polytopes $x \cup g$.

As $S \subseteq V$, we may conclude that in this configuration, every CF-family clearly does so, and as we see that $a_g < |S^g| \leq |S^u| < a_{g+1}$, it is indeed a CF-family in gap $g$.

Let us now take a look at the convex hulls of the incidence vectors. We observe that $\{S^g \setminus S^u\} \subseteq \{a_g - |I|\}$ for all entries $a_j$. Therefore, $F_j$ lies on the hyperplane $\{x \in \mathbb{R}^n : \Gamma^i x = a_j\}$. As $a_g \neq a_{g+1}$, we may conclude that aff$(F_g)$ and aff$(F_{g+1})$ do not intersect.

Let $U_g, U_{g+1}$ be the linear subspaces associated with the affine hulls aff$(F_g)$ and aff$(F_{g+1})$, respectively. If we can show that $U_g \cap U_{g+1} = \{0\}$, we may conclude that $F_g$ and $F_{g+1}$ lie in skew affine subspaces of $\mathbb{R}^n$ and thus conv$(F_g \cup F_{g+1}) \simeq F_g \ast F_{g+1}$.

As all the sets in $V_g$ are contained in $S^g$, all points of $F_g$ have $x_k = 0$ for all entries $k \in [n] \setminus S^g$. On the other hand, all the sets in $V_{g+1}$ contain $S^u$, so all points of $F_{g+1}$ have $x_k = 1$ for all entries $k \in S^u$. In either case, the named entries are identical for all points of the polytope, therefore $U_g \subseteq \bigcap_{k \in [n] \setminus S^g} x_k = 0$ and $U_{g+1} \subseteq \bigcap_{k \in S^u} x_k = 0$.

As $S^g \subseteq S^u$ implies $[n] \setminus S^u \cup S^u = [n]$, we may combine the two statements to find that $U_g \cap U_{g+1} \subseteq \bigcap_{k \in [n]} \{x_k = 0\} = \{0\}$, which we wanted to show.

We know the dimension of a free join of polytopes $F_g \ast F_{g+1}$ to be $\dim(F_g) + \dim(F_{g+1}) + 1$, therefore

$$d(n; a; \emptyset; \emptyset; S) = \dim(P(|S^g| - |I|; (a_g - |I|))) + \dim(P(n - |S^u| - |O|; (a_{g+1} - |S^u|))) + 1 = (|S^g| - |I| - 1) + (n - |S^u| - |O| - 1) + 1 = n - |I| - |O| + |S^g| - |S^u| - 1.$$
Firstly, we take a look at those classes where there is at least one layer on which the vertex set has at least two elements. This holds for the cases 5-9, i.e. many-1-pyramids, 1-many-pyramids, full-1-layer and full-2-layer sets generated by non-empty \( S \). In all those cases, we saw a representation of at least one of the portions of the vertex set in terms of our established Y-notation. This guarantees that no two tuples of input variables, in the condensed notation of \( S \) as a one- or two-element set which we introduced for each case, may yield the same vertex sets as any two condensed notations yielding the same bound for the set in Y-notation are either equal or yield a different second portion of the vertex set.

Secondly, let us have a look at all the cases in which there are vertices on exactly two layers and at least one of the portions can be represented using the Y-notation, i.e. cases 7-9 concerning 1-many-pyramids, many-1-pyramids and full-2-layer sets. We recall that any vertex set \( V \) of one of these classes generated by a configuration with empty \( S \) defined its vertices \( X \) by the same restrictions of the form \( A \subseteq X \subseteq B \) for both portions, i.e. for sets \( X \in V \) on both layers. Let us now have a look at a vertex set \( V' \) generated by a configuration with non-empty \( S \), that agrees with \( V \) on one of the two layers where they have at least two vertices. All the vertices of \( V \) on the other layer do satisfy \( A \subseteq X \subseteq B \), while in each of the cases 7-9, the vertices of \( V' \) on the other layer do not satisfy them, as one can readily check with the help of the reformulations provided by Proposition 54. Therefore, no two sets \( V = V(n; a; \emptyset; \emptyset; I; \emptyset; S) \) and \( V' = V(n; a; \emptyset; I'; \emptyset; O'; S) \) with some \( S \neq \emptyset \), can be equal. Thus, we may in fact partition the vertex sets in each of the classes of many-1-pyramids, 1-many-pyramids and full-2-layer sets, into those vertex sets generated by a configuration with empty \( S \) and those generated by a configuration with non-empty \( S \).

With this in mind, we may conclude our analysis of the vertex sets and do what we prepared to do throughout the last few sections: We will count the faces of \( P(n; a) \) by their dimension for all \( n \in \mathbb{N} \) and all cardinality sequences \( a \).

5 Counting the Faces

5.1 The f-Vector

The common notation to indicate the number of a polytope’s faces of a certain dimension is the f-vector, as defined in (Ziegler, 1995):

**Definition 55.** The f-vector of a d-polytope \( P \) is the vector

\[
f(P) = (f_{d-1}, f_0, f_1, \ldots, f_{d-1}) \in \mathbb{N}^{d+1},
\]

where \( f_d \) denotes the number of d-dimensional faces of \( P \).

Browsing through all the classes we introduced in Definition 44 and counting the number of such vertex sets generating a face of a certain dimension \( d \), we are now able to provide the number of faces of any given dimension.

**Theorem 56 (The f-Vector of \( P(n; a) \)).** Let \( n \in \mathbb{N} \) and \( a \) be a cardinality sequence for \( n \). The f-vector of \( P(n; a) \) is determined by

\[
f_{d-1} = 1, \quad f_0 = \sum_{i=1}^{m} \binom{n}{a_i}.
\]
while for $1 \leq d \leq \dim(P(n; a))$, $f_d$ is the coefficient of $X^d$ in the polynomial $F_{n,a}(X)$ defined as follows:

$$F_{n,a}(X) := \sum_{\ell=1}^{m-1} \sum_{g=0}^{a_\ell-1} \sum_{q=1+a_g}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p-1} \left[ - \sum_{\ell=2}^{m} \sum_{q=a_{\ell-1}+1}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p-1} \
+ \sum_{g=1}^{m-1} \frac{n}{a_g} \left( \frac{n-a_g}{a_{g+1}-a_g} \right) X^1 \
+ \sum_{g=1}^{m-1} \sum_{p=1+a_g}^{n} \sum_{q=1+a_g}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p} \
+ \sum_{g=1}^{m-1} \sum_{p=1+a_g}^{n} \sum_{q=1+a_g}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p} \
+ \sum_{r=1+a_g+1}^{n} \sum_{s=1+a_g+1}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p} \
+ \sum_{\ell=1}^{m-1} \sum_{q=1+a_{\ell+1}}^{n} \binom{n}{p} \binom{n-a_{\ell}}{q-a_{\ell}} X^{q-a_{\ell}} \
+ \sum_{k=2}^{m} \sum_{g=0}^{a_{\ell+1}} \sum_{p=0}^{a_{g+2}} \binom{n}{p} \binom{n-p}{q-p} X^{q-p} \
+ \sum_{g=1}^{m-2} \sum_{p=1+a_g}^{n} \sum_{q=1+a_g+2}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p}. \right]$$

**Proof.** We first recall that Proposition 37 states that the faces of $P(n; a)$ are exactly the convex hulls of incidence vectors of sets $V((n; a; M; I; O; S))$. Therefore counting the number of faces of a certain dimension is equivalent to counting distinct vertex sets yielding faces of a certain dimension.

For dimensions $-1$ and $0$, this is trivial: The only $(-1)$-dimensional face is the empty face, so $f_{-1} = 0$. The 0-dimensional faces of $P(n; a)$ are its vertices, we know them to be the incidence vectors of the sets in $C(n; a)$, of which we know there are $\sum_{i=1}^{m} \binom{n}{a_i}$. Let us now move on to the more interesting dimensions $d \geq 1$.

Each row of the polynomial in the theorem represents the faces generated by some configuration of vertex sets. We will go through all the classes examined in the previous sections and derive the polynomial from what we know. For each class, we shall name all the configurations of vertex sets we have seen to belong to this class. We will then count the number of faces of each dimension for each class and show that the numbers match with the coefficients of $X^d$ in the polynomial.

Let us handle the classes by the order in which they were introduced in Definition 44 and start with the full-1-layer sets.

We saw all 1-layer-sets for layer $\ell$ to be of the form $\{X \in \binom{[n]}{a_\ell} : A \subseteq X \subseteq B \}$ for sets $A \subset B \subseteq [n]$ with $|A| < a_\ell < |B|$, having dimension $|B| - |A| - 1$. We did however encounter different preconditions for $A$ and $B$: In the configuration of empty $S$ and either $M = \{ \min \}$ or $M = \{ \max \}$, there were no further restrictions, but these configurations do only yield vertex sets for layers 1 and $m$, respectively. In the configuration of empty $M$ and $S$, sets $A$ and $B$ were
required to be in gap $\ell - 1$ and $\ell$. In the configuration of empty $\mathcal{M}$ and $S$ a CF-family in gap $\ell - 1$, we found all constellations with $A$ in gap $\ell - 1$ and $a_\ell < |B|$, while for a CF-family in gap $\ell$, all constellations with $|A| < a_\ell$ and $B$ in gap $\ell$. As CF-families may only exist for gaps $g \in [m - 1]$, we can put this all together and say that we have the following distinct 1-layer-sets for layer $\ell \in \{1, m\}$:

$$\{(A, B) : A \subset B \subseteq [n], |A| < a_\ell < |B|\}$$

and the following distinct 1-layer-sets for layer $\ell \in \{2, \ldots, m - 1\}$:

$$\{(A, B) : A \subset B \subseteq [n], |A| < a_\ell < |B|\} \setminus \{(A, B) : A \subset B \subseteq [n], |A| \leq a_{\ell - 1}, a_{\ell + 1} \leq |B|\}$$

We may deduce from Proposition 48 that each vertex set generated by such a pair $(A, B)$ is unique and we may therefore determine the cardinality of the set by counting the number of such pairs. For every such pair, we will add up the monomial $X^{|B|-|A|-1}$, as that is the dimension of the associated face. We define $p$ to be a variable for the cardinality of $A$ and $q$ to be a variable for the cardinality of $B$. Then:

$$|\{(A, B) : A \subset B \subseteq [n], |A| < a_\ell < |B|\}| = \sum_{p=0}^{-1+a_\ell} \sum_{q=1+a_\ell}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p-1}$$

$$|\{(A, B) : A \subset B \subseteq [n], |A| \leq a_{\ell - 1}, a_{\ell + 1} \leq |B|\}| = \sum_{p=0}^{a_{\ell - 1}} \sum_{q=a_{\ell + 1}}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p-1}$$

We may now sum up the first cardinality for all layers and subtract the second cardinality for each layer $\ell \in \{2, \ldots, m - 1\}$. The result then is the portion of the polynomial counting the faces generated by 1-layer-sets:

$$\sum_{\ell=1}^{m} \sum_{p=0}^{-1+a_\ell} \sum_{q=1+a_\ell}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p-1} - \sum_{\ell=2}^{m-1} \sum_{p=0}^{a_{\ell - 1}} \sum_{q=a_{\ell + 1}}^{n} \binom{n}{p} \binom{n-p}{q-p} X^{q-p-1}$$

We have now already covered any vertex set originating from a configuration with $\mathcal{M} \neq \emptyset$. Let us now move on to the next class: The 1-1-pyramids. We saw they only occur if $\mathcal{M} = \emptyset$. For the configuration where $S$ is non-empty, we showed that any vertex set can be equally described as one with empty $S$. Therefore, we only need to count those sets for the configuration where $\mathcal{M}$ and $S$ are both empty. We saw that they occur exactly for sets $A \subseteq B \subseteq [n]$ with $|A| = a_g, a_{g+1} = |B|$ for some gap $g \in [m - 1]$. We may count the number of such combinations and include the monomial $X$, as we know the definition of the face of any 1-1-pyramid set to be exactly 1.

$$|\{(A, B) : A \subset B \subseteq [n], |A| = a_g, a_{g+1} = |B|, g \in [m - 1]\}| = \sum_{g=1}^{m-1} \binom{n}{a_g} \binom{n-a_g}{a_{g+1}-a_g} X^1$$

So far, so good. For the remaining cases (1-many and many-1-pyramids, full-2-layer and many-layer sets), thanks to Proposition 48 we saw that no two of them, in the condensed way we described them, are equal. We may therefore count them independently, starting with those vertex sets where $S$ is a CF-family in gap $g \in [m - 1]$, before working on those where $S = \emptyset$.

We first consider the 1-many-pyramids in gap $g$. They have a single set on layer $g$, denoted by $X_g$, while the vertices on layer $g + 1$ are uniquely determined by sets $S^{-1}$ and $\emptyset$, where $X_g \subset$
The dimension of the associated face is $|S^U| < |S^U| < a_{g+1} < n - |\mathcal{O}|$. We count them in the familiar way, using $A$ and $B$ as variables for the set around $a_{g+1}$ and multiply with the monomial $X^{[B] - |A|}$:

$$|\{(C, A, B) : C \subset A \subset B \subseteq [n], |C| = a_g < |A| < a_{g+1} < |B|, g \in [m - 1]\}|$$

$$= \sum_{g=1}^{m-1} \sum_{a_g}^{1} \sum_{q=1+a_g}^{n} \sum_{p=0}^{n} \left( n \atop a_g \right) \left( n - a_g \atop p - a_g \right) \left( n - q \atop q - p \right) X^{q-p}$$

The many-1-pyramids are very similar. Again, we have one set inside gap $g$, but this time there is one on layer $g + 1$ and one with cardinality smaller that $a_g$. We shall use the variables $A \subseteq B$ for the flexible ones and $C$ for the one set to be of cardinality $a_{g+1}$. The associated monomial is $X^{[B] - |A|}$ once more.

$$|\{(A, B, C) : C \subset A \subset B \subseteq [n], |A| < a_g < |B| < a_{g+1} = |C|, g \in [m - 1]\}|$$

$$= \sum_{g=1}^{m-1} \sum_{a_g}^{1} \sum_{q=1+a_g}^{n} \sum_{r=q}^{n} \left( n \atop p \right) \left( n - p \atop q - p \right) \left( n - q \atop a_{g+1} - q \right) X^{q-p}$$

The last class for $S$ a CF-family in a gap is that of full-2-layer sets, probably the most interesting configuration due to the join-property of the associated faces. Here, we have four sets $A \subset B \subseteq C \subset D$ arranged around gap $g \in [m - 1]$, where $|A| < a_g < |B| \leq |C| < a_{g+1} < |D|$. The dimension of the associated face is $|D| - |C| + |B| - |A| - 1$. We count as follows:

$$|\{(A, B, C, D) : A \subset B \subseteq C \subset D \subseteq [n], |A| < a_g < |B| \leq |C| < a_{g+1} < |D|, g \in [m - 1]\}|$$

$$= \sum_{g=1}^{m-1} \sum_{a_g}^{1} \sum_{q=1+a_g}^{n} \sum_{s=q}^{n} \sum_{r=s}^{n} \left( n \atop p \right) \left( n - p \atop q - p \right) \left( n - q \atop a_{g+1} - q \right) X^{q-p}$$

We have yet to cover those vertex sets for empty $\mathcal{M}$ and $\partial$ with vertices on at least two different layers which are not 1-1-pyramids. We know they are all of the form $\{X \in C(n, a) : A \subseteq X \subseteq B\}$. Let us first handle those where $A$ is on a layer, then those where $A$ is in a gap. We saw in the discussion of these configurations that the former case yields 1-many-pyramids and some many-layer sets, while the latter case yields many-1-pyramids, full-2-layer sets and the remaining many-layer sets.

If $A$ is on a layer $\ell$, we see that $\ell \leq m - 1$, otherwise the vertex set could not contain anything more than the set $A$ itself. Moreover, we see that for $B$, the inequality $a_{\ell+1} < |B|$ has to hold in order to contain more vertices than a mere 1-1-pyramid. On the other hand, as soon as these conditions are met, the resulting vertex set is guaranteed to be a 1-many-pyramid or a many-layer-set, as tells us Table 2. For both these classes, the associated face is of dimension $|B| - |A|$. Therefore, let us count all those configurations:

$$|\{(A, B) : A \subset B \subseteq [n], |A| = a_\ell, \ell \in [m - 1], a_{\ell+1} < |B|\}|$$

$$= \sum_{\ell=1}^{m-1} \sum_{q=1+a_\ell}^{n} \left( n \atop a_\ell \right) \left( n - a_\ell \atop q - a_\ell \right) X^{q-a_\ell}$$

What is left are those sets $\{X \in C(n, a) : A \subseteq X \subseteq B\}$ where $A$ is in a gap. If that gap is gap 0, the vertex set spreads across less than two layers if and only if $|B| < a_2$. 

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So we will only consider $B$'s with $a_2 \leq |B|$, for which we need to make sure that $m \geq 2$. Once more, the dimension of the associated face is $|B| - |A|$. 

\[
|\{(A, B) : A \subset B \subseteq [n], |A| < a_1, a_2 \leq |B|\}| = \sum_{k=2}^{m} \sum_{g=0}^{k-m-1+a_{g+1}} \sum_{p=0}^{n} \sum_{q=a_{g+2}}^{n} \binom{n}{p} \binom{n-p}{q-p} X^q p
\]

If, on the other hand, that gap is gap $g \in [m]$, again we need that $a_{g+2} \leq |B|$ in order to have at least two layers included. Therefore we count only those vertex sets where $g \leq m - 2$. We do so in the following way:

\[
|\{(A, B) : A \subset B \subseteq [n], g \in [m - 2], a_g < |A| < a_{g+1}, a_{g+2} \leq |B|\}| = \sum_{g=1}^{m-2} \sum_{p=1+a_g}^{n} \sum_{q=a_{g+2}}^{n} \binom{n}{p} \binom{n-p}{q-p} X^q p
\]

This concludes our analysis of all classes and configurations. The coefficient of $X^d$ in each of the parts of the polynomial is equivalent to the number of $d$-dimensional faces of that type. As we covered all classes, configurations and types of faces, the sum of those coefficients is equal to the number of $d$-dimensional faces of $P(n; a)$.

5.2 An Example

![Figure 8: C(5; (1, 4))](image)

Now that we learned about the $f$-vector for general $n \in \mathbb{N}$ and cardinality sequence $a$, let us apply our findings to determine the $f$-vector of the polytope $P(5; (1, 4))$ associated with our exemplary cardinality homogeneous set system $C(5; (1, 4))$.

We first observe that we can directly compute the values $f_{-1} = 1$ and $f_0 = \binom{5}{1} + \binom{5}{4} = 10$. For the rest, let’s apply what we know to the polynomial: $m = 2$ and thus all the sums over all $g \in [m - 1]$ only have the value $g = 1$, for which $a_g = 1$ and $a_{g+1} = 4$.

\[
F_{5(1,4)}(X) = \sum_{\ell=1}^{2} \sum_{p=0}^{-1+a_{\ell}} \sum_{q=1+a_{\ell}}^{5} \binom{5}{p} \binom{5-p}{q-p} X^{q-p-1} - \sum_{\ell=2}^{1} \ldots
\]
We may now simplify all the rows as far as possible and calculate the resulting parts of the
polynomial one by one. As each line represents some specific faces, we may read off the number
of such faces as well.

\[ F_{5; (1, 4)}(X) = 2 \cdot (10X^1 + 10X^2 + 5X^3 + 1X^4) \quad \text{(full-1-layer on layer 1 or 2)} \]
\[ + 20X^1 \quad \text{(1-1-pyr's)} \]
\[ + 30X^2 + 20X^3 \quad \text{(1-many-pyr's, } S \neq \emptyset) \]
\[ + 30X^2 + 20X^3 \quad \text{(many-1-pyr's, } S \neq \emptyset) \]
\[ + 30X^3 + 20X^4 \quad \text{(full-2-layer, } S \neq \emptyset) \]
\[ + 5X^4 \quad \text{(1-many-pyr's, } S = \emptyset) \]
\[ + 5X^4 + 1X^5 \quad \text{(many-1-pyr's, } S = \emptyset) + P(5; (1, 4)) \]
\[ = 40X^1 + 80X^2 + 80X^3 + 32X^4 + 1X^5 \]

We conclude that the f-vector of \( P(5; (1, 4)) \) is the following:

\[ f = (1, 10, 40, 80, 80, 32). \]

6 Conclusion

We have studied cardinality homogeneous set systems and the polytopes associated with them. We
showed how the bounding hyperplanes of the polytope intersect and provided a universal notation
to denote the vertex sets of their intersection. We provided a classification of the faces and an
analysis of their respective combinatorial type and dimension. Ultimately, the f-vector for any
polytope of a cardinality homogeneous set system has been provided.

It is desirable that some of the findings or approaches contained in this work lead to further
advancements in studying this fascinating class of polytopes. A more thorough understanding
of the structure of the polytope might facilitate the development of algorithms for combinatorial
problems related to it. As the description of all the faces of the polytope contains a description of
all the edges, concentrating on the polytope’s graph and its properties could be considered another possible topic for future research.

References


Eigenständigkeitserklärung


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